and

$$\rho^2 \dot{\varphi} = \frac{1}{\Delta}[2aMrE + (\rho^2 - 2Mr)L_z \operatorname{cosec}^2\theta]. \tag{163}$$

It is now clear that the problem of solving the equations of geodesic motion has been reduced to one of quadratures.

An alternative form of equation (161) which we shall find useful is

$$\rho^4 \dot{\theta}^2 = [K - (L_z - aE)^2] - [a^2(\delta_1 - E^2) + L_z^2 \operatorname{cosec}^2\theta]\cos^2\theta. \tag{164}$$

(a) *The separability of the Hamilton–Jacobi equation and an alternative derivation of the basic equations*

As we have stated, the existence of a fourth quantity that is conserved along a geodesic was first discovered by Carter by explicitly demonstrating the separability of the Hamilton–Jacobi equation. At the time, it was wholly unexpected; and it suggested that the other equations of mathematical physics might be similarly separable. Indeed, they were all eventually separated as we shall see in Chapters 8, 9, and 10. As the first of the chain of remarkable properties that characterize Kerr geometry, it is useful to follow Carter's demonstration of the separability of the Hamilton–Jacobi equation.

The Hamilton–Jacobi equation governing geodesic motion in a space-time with the metric tensor g^{ij} is given by

$$2\frac{\partial S}{\partial \tau} = g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}, \tag{165}$$

where S denotes Hamilton's principal function. With g^{ij} for the Kerr geometry given in Chapter 6, equation (135), equation (165) becomes

$$2\frac{\partial S}{\partial \tau} = \frac{\Sigma^2}{\rho^2\Delta}\left(\frac{\partial S}{\partial t}\right)^2 + \frac{4aMr}{\rho^2\Delta}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \varphi} - \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2. \tag{166}$$

It is convenient to rewrite this equation in the alternative form

$$2\frac{\partial S}{\partial \tau} = \frac{1}{\rho^2\Delta}\left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \varphi}\right]^2 - \frac{1}{\rho^2\sin^2\theta}\left[(a\sin^2\theta)\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \varphi}\right]^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2. \tag{167}$$

Assuming that the variables can be separated, we seek a solution of equation (167) of the form

$$S = \tfrac{1}{2}\delta_1 \tau - Et + L_z\varphi + S_r(r) + S_\theta(\theta), \tag{168}$$

where, as the notation indicates, S_r and S_θ are functions only of the variable specified. For the chosen form of S, equation (167) becomes

$$\delta_1 \rho^2 = \frac{1}{\Delta}[(r^2+a^2)E - aL_z]^2 - \frac{1}{\sin^2\theta}(aE\sin^2\theta - L_z)^2 - \Delta\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - \left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2. \tag{169}$$

With the aid of the identity

$$(aE\sin^2\theta - L_z)^2\operatorname{cosec}^2\theta = (L_z^2\operatorname{cosec}^2\theta - a^2E^2)\cos^2\theta + (L_z - aE)^2, \tag{170}$$

we can rewrite equation (169) in the form

$$\left\{\Delta\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - \frac{1}{\Delta}[(r^2+a^2)E - aL_z]^2 + (L_z - aE)^2 + \delta_1 r^2\right\} + \left\{\left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2 + (L_z^2\operatorname{cosec}^2\theta - a^2E^2)\cos^2\theta + \delta_1 a^2\cos^2\theta\right\} = 0. \tag{171}$$

The separability of the equation is now manifest and we infer that

$$\Delta\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 = \frac{1}{\Delta}[(r^2+a^2)E - aL_z]^2 - [\mathscr{Q} + (L_z - aE)^2 + \delta_1 r^2] \tag{172}$$

and

$$\left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2 = \mathscr{Q} - (L_z^2\operatorname{cosec}^2\theta - a^2E^2 + \delta_1 a^2)\cos^2\theta, \tag{173}$$

where $\mathscr{Q}$ is a separation constant. With the abbreviations

$$R = [(r^2+a^2)E - aL_z]^2 - \Delta[\mathscr{Q} + (L_z - aE)^2 + \delta_1 r^2] \tag{174}$$

and

$$\Theta = \mathscr{Q} - [a^2(\delta_1 - E^2) + L_z^2\operatorname{cosec}^2\theta]\cos^2\theta, \tag{175}$$

the solution for S is

$$S = \tfrac{1}{2}\delta_1\tau - Et + L_z\varphi + \int^r \frac{\sqrt{R(r)}}{\Delta}\,\mathrm{d}r + \int^\theta \mathrm{d}\theta\sqrt{\Theta(\theta)}. \tag{176}$$

The basic equations governing the motion can be deduced from the solution (176) for the principal function S by the standard procedure of setting to zero the partial derivatives of S with respect to the different constants of the motion—$\mathscr{Q}$, δ_1, E, and L_z in this instance. Thus, we find that

$$\frac{\partial S}{\partial \mathscr{Q}} = \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial \mathscr{Q}}\,\mathrm{d}r + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial \mathscr{Q}}\,\mathrm{d}\theta = 0 \tag{177}$$

leads to the equation

$$\int^r \frac{dr}{\sqrt{R}} = \int^\theta \frac{d\theta}{\sqrt{\Theta}}. \tag{178}$$

Similarly, we find

$$\tau = \int^r \frac{r^2}{\sqrt{R}}\,dr + a^2 \int^\theta \frac{\cos^2\theta}{\sqrt{\Theta}}\,d\theta, \tag{179}$$

$$\begin{aligned} t &= \frac{1}{2}\int^r \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial E}\,dr + \frac{1}{2}\int^\theta \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial E}\,d\theta \\ &= \tau E + 2M\int^r r[r^2E - a(L_z - aE)]\frac{dr}{\Delta\sqrt{R}}, \end{aligned} \tag{180}$$

and

$$\begin{aligned} \varphi &= -\frac{1}{2}\int^r \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial L_z}\,dr - \frac{1}{2}\int^\theta \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial L_z}\,d\theta \\ &= a\int^r [(r^2+a^2)E - aL_z]\frac{dr}{\Delta\sqrt{R}} + \int^\theta (L_z \operatorname{cosec}^2\theta - aE)\frac{d\theta}{\sqrt{\Theta}}, \end{aligned} \tag{181}$$

where simplifications have been effected with the aid of equation (178).

It can now be verified that equations (178)–(181) are entirely equivalent to the set (160)–(163) with the identification

$$\mathscr{Q} = K - (L_z - aE)^2. \tag{182}$$

In particular, with this identification, the right-hand sides of equations (160) and (164) agree with the present definitions of R and Θ in equations (174) and (175); and the relation (178) is an immediate consequence of equations (160) and (164). Our basic equations, then, are

$$\rho^4\dot{r}^2 = R; \quad \rho^4\dot{\theta}^2 = \Theta, \tag{183}$$

and

$$\left.\begin{aligned} \rho^2\dot{\varphi} &= \frac{1}{\Delta}[2aMrE + (\rho^2 - 2Mr)L_z \operatorname{cosec}^2\theta], \\ \rho^2\dot{t} &= \frac{1}{\Delta}(\Sigma^2 E - 2aMrL_z), \end{aligned}\right\} \tag{184}$$

where

$$R = E^2r^4 + (a^2E^2 - L_z^2 - \mathscr{Q})r^2 + 2Mr[\mathscr{Q} + (L_z - aE)^2] - a^2\mathscr{Q} - \delta_1 r^2\Delta \tag{185}$$

and

$$\Theta = \mathscr{Q} + (a^2E^2 - L_z^2\operatorname{cosec}^2\theta)\cos^2\theta - \delta_1 a^2\cos^2\theta. \tag{186}$$

It is convenient to assemble in one place the various formulae giving the tensor and the tetrad components of the four momentum. We have

$$\left.\begin{aligned} -p_r &= \frac{\rho^2}{\Delta}p^r = \frac{\sqrt{R}}{\Delta}, \quad -p_\theta = \rho^2 p^\theta = \sqrt{\Theta}, \\ p_\varphi &= \frac{2aMr\sin^2\theta}{\rho^2}p^t - \left(r^2 + a^2 + \frac{2a^2Mr}{\rho^2}\sin^2\theta\right)(\sin^2\theta)p^\varphi = -L_z, \\ p_t &= \left(1 - \frac{2Mr}{\rho^2}\right)p^t + \frac{2aMr\sin^2\theta}{\rho^2}p^\varphi = E, \end{aligned}\right\} \qquad (187)$$

and

$$\left.\begin{aligned} p^{(t)} &= p_{(t)} = e^{-\nu}(E - \omega L_z) = e^{+\nu}p^t, \\ p^{(r)} &= -p_{(r)} = -e^{-\mu_2}p_r = +e^{+\mu_2}p^r, \\ p^{(\theta)} &= -p_{(\theta)} = -e^{-\mu_3}p_\theta = e^{+\mu_3}p^\theta, \\ \text{and} \quad p^{(\varphi)} &= -p_{(\varphi)} = e^{\psi}p^\varphi = -e^{-\psi}p_\varphi = e^{-\psi}L_z. \end{aligned}\right\} \qquad (188)$$

63. The null geodesics

In our considerations of the general non-planar orbits, we shall concentrate on delineating the projection of the orbits on to the (r, θ) plane: the variations of t and φ along the orbits do not reveal any special features that have not already been displayed by the planar orbits on the equatorial plane.

For the null geodesics $\delta_1 = 0$, and it is convenient to minimize the number of parameters by letting

$$\xi = L_z/E \qquad \text{and} \qquad \eta = \mathscr{Q}/E^2, \qquad (189)$$

and writing R and Θ in place of R/E^2 and Θ/E^2:

$$R = r^4 + (a^2 - \xi^2 - \eta)r^2 + 2M[\eta + (\xi - a)^2]r - a^2\eta \qquad (190)$$

and

$$\Theta = \eta + a^2\cos^2\theta - \xi^2\cot^2\theta. \qquad (191)$$

The two parameters ξ and η replace the single impact parameter, D, by which we distinguished the null geodesics in the equatorial plane. The parameters ξ and η are in fact related very simply to the 'celestial coordinates' α and β of the image as seen by an observer at infinity who receives the light ray. Making use of the expressions (188), we readily verify that

$$\left.\begin{aligned} \alpha &= \left(\frac{rp^{(\varphi)}}{p^{(t)}}\right)_{r\to\infty} = \xi\,\mathrm{cosec}\,\theta_0 \\ \text{and} \quad \beta &= \left(\frac{rp^{(\theta)}}{p^{(t)}}\right)_{r\to\infty} = (\eta + a^2\cos^2\theta_0 - \xi^2\cot^2\theta_0)^{1/2} = -p_{\theta_0}. \end{aligned}\right\} \qquad (192)$$