

HIGHER ORDER ALMOST AUTOMORPHY, RECURRENCE SETS AND THE REGIONALLY PROXIMAL RELATION

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ABSTRACT. In this paper, d -step almost automorphic systems are studied for $d \in \mathbb{N}$, which are the generalization of the classical almost automorphic ones.

For a minimal topological dynamical system (X, T) it is shown that the condition $x \in X$ is d -step almost automorphic can be characterized via various subsets of \mathbb{Z} including the dual sets of d -step Poincaré and Birkhoff recurrence sets, and Nil_d Bohr₀-sets by considering $N(x, V) = \{n \in \mathbb{Z} : T^n x \in V\}$, where V is an arbitrary neighborhood of x . Moreover, it turns out that the condition $(x, y) \in X \times X$ is regionally proximal of order d can also be characterized via various subsets of \mathbb{Z} including d -step Poincaré and Birkhoff recurrence sets, SG_d sets, the dual sets of Nil_d Bohr₀-sets, and others by considering $N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$, where U is an arbitrary neighborhood of y .

1. INTRODUCTION

In the past few decades, it has become apparent both in ergodic theory and additive combinatorics that nilpotent groups and a higher order Fourier analysis play an important role. In this paper we will apply results obtained by the same authors in [32] to study higher order automorphic systems, namely d -step almost automorphic systems which by the definition are the almost one-to-one extensions of their maximal d -step nilfactors. Since for a minimal system the maximal d -step nilfactor is induced by the regionally proximal relation of order d (which is a closed invariant equivalence relation [28, 36]), the natural way we study d -step almost automorphic systems is that we first get some characterizations of regionally proximal relation of order d , and then obtain results for d -step almost automorphic systems. In the process doing above many interesting subsets of \mathbb{Z} including higher order Poincaré and Birkhoff recurrence sets (usual and cubic versions), higher order Bohr sets, SG_d sets (introduced in [27]) and others are involved. In this section we introduce the background and state the main results of the paper.

Date: Sept. 30, 2011.

2000 Mathematics Subject Classification. Primary: 37B05, 37B20.

Key words and phrases. almost automorphy, nilsystem, Poincaré recurrence set, Birkhoff recurrence set, the regionally proximal relation of order d .

Huang is supported by NNSF of China (10911120388), Fok Ying Tung Education Foundation and the Fundamental Research Funds for the Central Universities, Shao is supported by NNSF of China (10871186) and Program for New Century Excellent Talents in University, and Huang+Ye are supported by NNSF of China (11071231).

1.1. **Background.** First we give some background.

1.1.1. *Almost periodicity and almost automorphy.* The study of (uniformly) almost periodic functions was initiated by Bohr in a series of three papers 1924-26 in [7]. The literature on almost periodic functions is enormous, and the notion has been generalized in several directions. Nowadays the theory of almost periodic functions may be recognized as the representation theory of compact Hausdorff groups: every topological group G has a group compactification $\alpha_G : G \rightarrow bG$ such that the space of almost periodic functions on G is just the set of all functions $f \circ \alpha_G$ with $f \in C(bG)$. The compactification (α_G, bG) of G is called the *Bohr compactification* of G .

Related to the almost periodic functions are the *almost automorphic functions*: these functions turn out to be the ones of the form $h \circ \alpha_G$ with h a bounded continuous function on $\alpha_G(G)$ (if h is uniformly continuous and bounded on $\alpha_G(G)$, then it extends to an $f \in C(bG)$, so $h \circ \alpha_G = f \circ \alpha_G$ is almost periodic on G).

The notion of almost automorphy was first introduced by Bochner in 1955 in a work of differential geometry [8, 9]. Taking G for the present to be the group of integers \mathbb{Z} and an almost automorphic function f has the property that from any sequence $\{n'_i\} \subseteq \mathbb{Z}$ one may extract a subsequence $\{n_i\}$ such that both

$$\lim_{i \rightarrow \infty} f(t + n_i) = g(t) \quad \text{and} \quad \lim_{i \rightarrow \infty} g(t - n_i) = f(t)$$

hold for each $t \in \mathbb{Z}$ and some function g , not necessarily uniformly. Bochner has observed that almost periodic functions are almost automorphic, but the converse is not true [9]. Veech showed that the almost automorphic functions can be characterized in terms of the almost periodic ones, and vice versa [38]. In the same paper, Veech considered the system associated with an almost automorphic function, and introduced the notion of *almost automorphic point* (*AA point*, for short) in topological dynamical systems (t.d.s. for short). For a t.d.s. (X, T) , a point $x \in X$ is said to be *almost automorphic* if from any sequence $\{n'_i\} \subseteq \mathbb{Z}$ one may extract a subsequence $\{n_i\}$ such that

$$\lim_{j \rightarrow \infty} \lim_{i \rightarrow \infty} T^{n_i - n_j} x = x.$$

Also Veech gave the structure theorem for minimal systems with an almost automorphic point: each minimal almost automorphic system is almost one-to-one extension of its maximal equicontinuous factor [38, 39].

Note that in [38] all works were done for general groups. The notion of almost automorphy is very useful in the study of differential equations, and see [37] and references there for more information on this topic.

1.1.2. *The equicontinuous structure relation S_{eq} , almost automorphy and Bohr₀ sets.* For a t.d.s. (X, T) , it was proved in [12] that there exists on X a closed T -invariant equivalence relation, S_{eq} , such that $(X/S_{eq}, T)$ is an equicontinuous system. S_{eq} is called the *equicontinuous structure relation*. It was also showed in [12] that S_{eq} is the

smallest closed T -invariant equivalence relation containing the regionally proximal relation $\mathbf{RP} = \mathbf{RP}(X)$ (recall that $(x, y) \in \mathbf{RP}$ if there are sequences $x_i, y_i \in X, n_i \in \mathbb{Z}$ such that $x_i \rightarrow x, y_i \rightarrow y$ and $(T \times T)^{n_i}(x_i, y_i) \rightarrow (z, z), i \rightarrow \infty$, for some $z \in X$). A natural question was whether $S_{eq} = \mathbf{RP}(X)$ for all minimal t.d.s.? Veech [39] gave the first positive answer to this question, i.e. he proved that $S_{eq} = \mathbf{RP}(X)$ for all minimal t.d.s. under abelian group actions. As a matter of fact, Veech proved that for a minimal t.d.s. $(x, y) \in S_{eq}$ if and only if there is a sequence $\{n_i\} \subset \mathbb{Z}$ and $z \in X$ such that

$$T^{n_i}x \longrightarrow z \quad \text{and} \quad T^{-n_i}z \longrightarrow y, \quad i \rightarrow \infty.$$

As a direct corollary, for a minimal t.d.s. (X, T) , a point $x \in X$ is almost automorphic if and only if

$$\mathbf{RP}[x] = \{y \in X : (x, y) \in \mathbf{RP}\} = \{x\}.$$

Also from Veech's approach, it is easy to show that for a minimal t.d.s. (X, T) , $(x, y) \in \mathbf{RP}$ if and only if for each neighborhood U of y , $N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$ contains some Δ -set¹. Hence it is not difficult to get another equivalent condition for almost automorphic point [16]: a point $x \in X$ is almost automorphic if and only if it is Δ^* -recurrent.²

Recall a subset $A \subseteq \mathbb{Z}$ is a *Bohr₀ set* if there exists an equicontinuous system (X, T) , a point $x_0 \in X$ and its open neighborhood U such that $N(x_0, U) = \{n \in \mathbb{Z} : T^n x_0 \in U\}$ is contained in A .³ Since every point in an equicontinuous system is almost automorphic, it follows that each Bohr₀ set is a Δ^* -set. The converse does not hold [4]. But Δ^* -set is not too far from being Bohr₀-set. It is showed by Host and Kra recently that each Δ^* -set is a piecewise Bohr₀-set, meaning that it agrees with a Bohr₀-set on a sequence of intervals whose lengths tend to infinity [27].

1.1.3. *Poincaré recurrence sets and almost automorphy.* Note that the set $N(U, U) = \{n \in \mathbb{Z} : U \cap T^{-n}U \neq \emptyset\}$ is very important in a dynamical system. Birkhoff recurrence theorem says that whenever (X, T) is a minimal t.d.s. and $U \subseteq X$ a nonempty open set, then $N(U, U) \neq \emptyset$. The measurable version of this phenomenon is the famous Poincaré's Recurrence Theorem: Let (X, \mathcal{X}, μ, T) be a measure preserving system and $A \in \mathcal{X}$ with $\mu(A) > 0$, then $N_\mu(A, A) = \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A) > 0\}$ is infinite.

In [16, 15] Furstenberg introduced the notion of Poincaré recurrence sets and Birkhoff recurrence sets. A subset P of \mathbb{Z} is called a *Poincaré recurrence set* (or a

¹A Δ -set is a set of differences $A - A = \{a - b : a, b \in A\}$ for some infinite subset $A \subset \mathbb{Z}$; and a Δ^* -set is a set that has nontrivial intersection with the set of $A - A$ from any infinite set A .

²Let \mathcal{F} be a collection of subsets of \mathbb{Z} and let (X, T) be a system. A point x of X is called \mathcal{F} -recurrent if $N(x, U) \in \mathcal{F}$ for all neighborhood U of x .

³There are lots of equivalent definitions for Bohr set. For example, one may define Bohr sets as follows: A subset $A \subseteq \mathbb{Z}$ is a *Bohr set* if there exist $m \in \mathbb{N}, \alpha \in \mathbb{T}^m$, and an open set $U \subseteq \mathbb{T}^m$ such that $\{n \in \mathbb{Z} : n\alpha \in U\}$ is contained in A ; the set A is a *Bohr₀ set* if additionally $0 \in U$. See [4, 33] for more details.

set of measurable recurrence) if whenever (X, \mathcal{X}, μ, T) is a measure preserving system and $A \in \mathcal{X}$ has positive measure, then $P \cap N_\mu(A, A) \neq \emptyset$. Similarly, a subset $P \subset \mathbb{Z}$ is called a *Birkhoff recurrence set* (or a *set of topological recurrence*) if whenever (X, T) is a minimal t.d.s. and $U \subseteq X$ a nonempty open set, then $P \cap N(U, U) \neq \emptyset$. Let \mathcal{F}_{Poi} and \mathcal{F}_{Bir} denote the collections of Poincaré and Birkhoff recurrence sets of \mathbb{Z} respectively.

In [31], it was showed for a minimal t.d.s. $(x, y) \in \mathbf{RP}$ if and only if for each neighborhood U of y , $N(x, U) \in \mathcal{F}_{Poi}$. Hence it is possible to use Poincaré recurrence set to get another equivalent condition for almost automorphic point: *a point $x \in X$ is almost automorphic if and only if it is \mathcal{F}_{Poi}^* -recurrent*, where \mathcal{F}_{Poi}^* is the collection of subsets intersecting all sets from \mathcal{F}_{Poi} . One has similar results for Birkhoff recurrence sets.

1.1.4. *Multiple ergodic averages and nilsystems.* It is stated by Von Neumann and Birkhoff ergodic theorems that ergodic average $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$ converges in L^2 and pointwisely respectively. The study of the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \dots f_d(T^{dn} x)$$

begins from the Furstenberg's beautiful proof of Szemerédi's theorem via ergodic theory [14] in the 1970's. After nearly 30 years' efforts of many researchers, this problem of L^2 case was finally solved by Host and Kra in [25] (see also Ziegler [42]). In their proofs the theory of nilsystems plays a great role. The structure theorem of [25, 42] states that if one wants to understand the multiple ergodic averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \dots f_d(T^{dn} x),$$

one can replace each function f_i by its conditional expectation on some $d - 1$ -step nilsystem (1-step nilsystem is the Kroneker's one). Thus one can reduce the problem to the study of the same average in a nilsystem.

The study of the topological correspondence of the multiple ergodic averages also has a long history. It maybe goes back to the study of the equicontinuous structure relation $S_{eq}(X)$ of a t.d.s. (X, T) in the 1960's, and more recently Glasner's work [20, 21] etc.. It turns out the notion of the regionally proximal relation of order d defined in [29, 28] plays an important role.

Definition 1.1. Let (X, T) be a t.d.s. and let $d \geq 1$ be an integer. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta, \rho(y, y') < \delta$, and

$$\rho(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') < \delta \text{ for any } \epsilon \in \{0, 1\}^d, \epsilon \neq (0, \dots, 0),$$

where $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d \epsilon_i n_i$. The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order d* .

It is easy to see that $\mathbf{RP}^{[d]}(X)$ is a closed and invariant relation for all $d \in \mathbb{N}$. When $d = 1$, $\mathbf{RP}^{[d]}(X)$ is nothing but the classical regionally proximal relation. In [28], for distal minimal t.d.s. the authors showed that $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation, and the quotient of X under this relation is its maximal d -step nilfactor. These results were proved to be true for general minimal t.d.s. [36].

1.1.5. *Nilsystems and nilsequences.* Furstenberg's proof of Szemerédi's theorem via ergodic theory paved the way for new combinatorial results via ergodic methods, as well as leading to numerous developments within ergodic theory. More recently, the interaction between the fields has taken a new dimension, with ergodic objects being imported into the finite combinatorial setting. Some objects at the center of this interchange are nilsequences and the nilsystems on which they are defined (see, for example, [5, 22, 23, 24, 25, 26, 27, 28]).

Nilsequences are defined by evaluating a function along the orbit of a point in the homogeneous space of a nilpotent Lie group. We recall the definition of a nilsequence. A *basic d -step nilsequence* is a sequence of the form $\{f(T^n x) : n \in \mathbb{Z}\}$, where $d \in \mathbb{N}$ and (X, T) is a basic d -step nilsystem, $f : X \rightarrow \mathbb{C}$ is a continuous function, and $x \in X$. A *d -step nilsequence* is a uniform limit of basic d -step nilsequences.

One can define a generalization of a Bohr₀ set [27]:

Definition 1.2. A subset $A \subseteq \mathbb{Z}$ is a *Nil_d Bohr₀-set* if there exist a d -step nilsystem (X, T) , $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that

$$\{n \in \mathbb{Z} : T^n x_0 \in U\}$$

is contained in A .

Denote by \mathcal{F}_{Bohr_0} and $\mathcal{F}_{d,0}$ the family generated by all Bohr₀-sets and Nil_d Bohr₀-sets respectively. Note that $\mathcal{F}_{Bohr_0} = \mathcal{F}_{1,0}$.

1.1.6. *d -step almost automorphy.* Similar to the definition of almost automorphy, now we have the definition of d -step almost automorphy for all $d \in \mathbb{N}$:

Definition 1.3. Let (X, T) be a minimal t.d.s. and $x \in X$, $d \in \mathbb{N}$. x is called *d -step almost automorphic* (or d -step AA for short) if $\mathbf{RP}^{[d]}[x] = \{x\}$. A minimal t.d.s. is called *d -step almost automorphic* if it has a d -step almost automorphic point.

Since $\mathbf{RP}^{[d]}$ is an equivalence relation for minimal t.d.s. [36], by definition it follows that

Proposition 1.4. *Let (X, T) be a minimal t.d.s.. Then (X, T) is a d -step almost automorphic system for some $d \in \mathbb{N}$ if and only if it is an almost one-to-one extension of its maximal d -step nilfactor.*

1.1.7. *Higher order recurrence sets.* In this paper, we will use recurrence sets to characterize d -step almost automorphy. First we need to generalize the recurrence sets to a higher order version.

Here is a generalization of Poincaré recurrence subsets [13]. Let $d \in \mathbb{N}$.

Definition 1.5. (1) We say that $S \subset \mathbb{Z}$ is a set of d -recurrence if for all measure preserving systems (X, \mathcal{X}, μ, T) and for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that

$$\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0.$$

(2) We say that $S \subset \mathbb{Z}$ is a set of d -topological recurrence if for every minimal t.d.s. (X, T) and for every nonempty open subset U of X , there exists $n \in S$ such that

$$U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset.$$

Let \mathcal{F}_{Poi_d} (resp. \mathcal{F}_{Bir_d}) be the family generated by the collection of all sets of d -recurrence (resp. sets of d -topological recurrence). It is obvious by definitions that $\mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d}$. It is showed in [32] that these sets are contained in the dual family of Nil_d -Bohr₀ sets.

Proposition 1.6. [32] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d} \subset \mathcal{F}_{d,0}^*,$$

where $\mathcal{F}_{d,0}^*$ is the collection of all sets intersecting all Nil_d Bohr₀ sets.

Note that $\mathcal{F}_{Poi} \neq \mathcal{F}_{Bir}$ [35]. Though we can not prove if $\mathcal{F}_{Bir_d} = \mathcal{F}_{d,0}^*$, we will show it is hard to distinguish them via dynamical methods (Theorem 1.8).

Remark 1.7. The above definitions are slightly different from the ones introduced in [13], namely we do not require $n \neq 0$. The main reason we define in this way is that for each $A \in \mathcal{F}_{d,0}$, $0 \in A$. Thus $\{0\} \cup C \in \mathcal{F}_{d,0}^*$ for each $C \subset \mathbb{Z}$.

1.2. **Main results.** Now we are ready to state the main results.

1.2.1. *Regionally proximal relation of order d and d -step almost automorphy.* The following theorem shows that we can use \mathcal{F}_{Poi_d} , \mathcal{F}_{Bir_d} and $\mathcal{F}_{d,0}^*$ to characterize regionally proximal pairs of order d .

Theorem 1.8. *Let (X, T) be a minimal t.d.s.. The following statements are equivalent:*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$.
- (2) $N(x, U) \in \mathcal{F}_{Poi_d}$ for each neighborhood U of y .
- (3) $N(x, U) \in \mathcal{F}_{Bir_d}$ for each neighborhood U of y .
- (4) $N(x, U) \in \mathcal{F}_{d,0}^*$ for each neighborhood U of y .

Using the Ramsey property of the families, we can show that one can use $\mathcal{F}_{Poi_d}^*$, $\mathcal{F}_{Bir_d}^*$ and $\mathcal{F}_{d,0}$ to characterize d -step almost automorphy.

Theorem 1.9. *Let (X, T) be a minimal t.d.s. and $d \in \mathbb{N}$. Then the following statements are equivalent*

- (1) (X, T) is d -step almost automorphic.
- (2) There is $x \in X$ such that $N(x, V) \in \mathcal{F}_{Poi_d}^*$ for each neighborhood V of x .
- (3) There is $x \in X$ such that $N(x, V) \in \mathcal{F}_{Boi_d}^*$ for each neighborhood V of x .
- (4) There is $x \in X$ such that $N(x, V) \in \mathcal{F}_{d,0}$ for each neighborhood V of x .

1.2.2. *d -step almost automorphy and SG_d -sets.* In this paper, we also discuss SG_d -sets introduced by Host and Kra recently [27] and show that one may use it to characterize regionally proximal pairs of order d .

Let $d \geq 1$ be an integer and let $P = \{p_i\}_i$ be a (finite or infinite) sequence in \mathbb{Z} . The set of sums with gaps of length less than d of P is the set $SG_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \dots + \epsilon_n p_n$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than d . A subset $A \subseteq \mathbb{Z}$ is an SG_d -set if $A = SG_d(P)$ for some infinite sequence of \mathbb{Z} ; and it is an SG_d^* -set if $A \cap SG_d(P) \neq \emptyset$ for every infinite sequence P in \mathbb{Z} . Let \mathcal{F}_{SG_d} be the family generated by all SG_d -sets. Note that each SG_1 -set is a Δ -set, and each SG_1^* -set is a Δ^* -set.

The following is the main result of [27]

Proposition 1.10 (Host-Kra). *Every SG_d^* -set is a $PW\text{-Nil}_d$ Bohr $_0$ -set.*

Host and Kra [27] asked the following

Question 1.11. *Is every Nil_d Bohr $_0$ -set an SG_d^* -set?*

We have

Theorem 1.12. *Let (X, T) be a minimal t.d.s. and $x, y \in X$, $d \in \mathbb{N}$. Then $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{SG_d}$ for each neighborhood U of y .*

Combining Theorems 1.8 and 1.12 we see that Nil_d Bohr $_0$ -sets and SG_d^* -sets are closely related. A direct corollary of Theorem 1.12 is: let (X, T) be a minimal t.d.s., $x \in X$, and $d \in \mathbb{N}$. If x is $\mathcal{F}_{SG_d}^*$ -recurrent, then it is d -step almost automorphic. We have the following conjecture.

Conjecture 1.13. *Let (X, T) be a minimal t.d.s., $x \in X$, and $d \in \mathbb{N}$. Then x is d -step almost automorphic if and only if it is SG_d^* -recurrent.*

Since SG_d -sets do not have the Ramsey property (Appendix A), we can not apply the methods in the proof of Theorem 1.9 to show the above conjecture. Note that if Question 1.11 has a positive answer, then by using Theorem 1.9 the above conjecture holds.

1.2.3. *Cubic version of multiple Poincaré recurrence sets.* One can also characterize the higher order regionally proximal relation via cubic version of multiple Poincaré recurrence sets. For $d \in \mathbb{N}$, a subset F of \mathbb{Z} is a *Poincaré recurrence set of order d* if for each measure preserving system (X, \mathcal{B}, μ, T) and $A \in \mathcal{B}$ with positive measure there are $n_1, \dots, n_d \in \mathbb{Z}$ such that $FS(\{n_i\}_{i=1}^d) = \{n_{i_1} + \dots + n_{i_k} : 1 \leq i_1 < \dots < i_k \leq d\} \subset F$ and

$$\mu\left(A \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A\right)\right) > 0.$$

Similarly, we define Birkhoff recurrence sets of order d . Let for $d \in \mathbb{N}$, \mathcal{F}_{P_d} (resp. \mathcal{F}_{B_d}) be the family of all Poincaré recurrence sets of order d (resp. the family of all Birkhoff recurrence sets of order d).

Via recurrence sets of order d , we have the following result:

Theorem 1.14. *Let (X, T) be a minimal t.d.s. and $x, y \in X$, $d \in \mathbb{N}$. Then the following statements are equivalent:*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$.
- (2) $N(x, U) \in \mathcal{F}_{P_d}$ for each neighborhood U of y .
- (3) $N(x, U) \in \mathcal{F}_{B_d}$ for each neighborhood U of y .

A direct corollary of Theorem 1.14 is: let (X, T) be a minimal t.d.s., $x \in X$, and $d \in \mathbb{N}$. If x is $\mathcal{F}_{P_d}^*$ -recurrent, or $\mathcal{F}_{B_d}^*$ -recurrent then it is d -step almost automorphic. We have the following conjecture.

Conjecture 1.15. *Let (X, T) be a minimal t.d.s., $x \in X$, and $d \in \mathbb{N}$. Then x is d -step almost automorphic if and only if it is $\mathcal{F}_{P_d}^*$ -recurrent if and only if it is $\mathcal{F}_{B_d}^*$ -recurrent.*

We note that there are two possible ways to show the conjecture: (1) prove \mathcal{F}_{P_d} and \mathcal{F}_{B_d} have the Ramsey property, (2) prove $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d} \subset \mathcal{F}_{d,0}^*$. Unfortunately, at this moments we can not prove neither of them.

1.3. **Organization of the paper.** We organize the paper as follows: in Section 2, we give the basic definitions and facts used in the paper. In Section 3, we study Nil_d -Bohr₀ sets and higher order recurrence sets, and use them to characterize $\mathbf{RP}^{[d]}$. In Section 4, we study SG_d sets and use them to characterize $\mathbf{RP}^{[d]}$. In Section 5, we introduce the cubic version of multiple recurrence sets, and also use them to characterize $\mathbf{RP}^{[d]}$. In the final section, we introduce the notion of d -step almost automorphy and obtain various characterizations. In the Appendix, we show SG_2 does not have the Ramsey property, Theorem 2.5 holds for general compact Hausdorff systems and the cubic version of the multiple Poincaré and Birkhoff recurrence sets can be interpreted using intersectiveness.

2. PRELIMINARIES

2.1. **Measurable and topological dynamics.** In this subsection we give some basic notions in ergodic theory and topological dynamics.

2.1.1. *Measurable systems.* In this paper, a *measure preserving system* is a quadruple (X, \mathcal{X}, μ, T) , where (X, \mathcal{X}, μ) is a Lebesgue probability space and $T : X \rightarrow X$ is an invertible measure preserving transformation.

We write $\mathcal{I} = \mathcal{I}(T)$ for the σ -algebra $\{A \in \mathcal{X} : T^{-1}A = A\}$ of invariant sets. A system is *ergodic* if all the T -invariant sets have measure either 0 or 1. (X, \mathcal{X}, μ, T) is *weakly mixing* if the product system $(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times \mu, T \times T)$ is ergodic.

2.1.2. *Topological dynamical systems.* A *transformation* of a compact metric space X is a homeomorphism of X to itself. A *topological dynamical system*, referred to more succinctly as just a t.d.s. or a *system*, is a pair (X, T) , where X is a compact metric space and $T : X \rightarrow X$ is a transformation. We use $\rho(\cdot, \cdot)$ to denote the metric on X .

A t.d.s. (X, T) is *transitive* if X is uncountable, and there exists some point $x \in X$ whose orbit $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X . Moreover, we call such a point a *transitive point*. The system is *minimal* if the orbit of any point is dense in X . This property is equivalent to saying that X and the empty set are the only closed invariant sets in X .

A *factor* of a t.d.s. (X, T) is another t.d.s. (Y, S) such that there exists a continuous and onto map $\phi : X \rightarrow Y$ satisfying $S \circ \phi = \phi \circ T$. In this case, (X, T) is called an *extension* of (Y, S) . The map ϕ is called a *factor map*.

2.1.3. We also make use of a more general definition of a measurable or topological system. That is, instead of just a single transformation T , we consider commuting homeomorphisms T_1, \dots, T_k of X or a countable abelian group of transformations.

2.2. **Cubes and faces.** In the following subsections, we will introduce notions about cubes, faces and dynamical parallelepipeds. For more details see [25, 28, 29].

2.2.1. Let X be a set, let $d \geq 1$ be an integer, and write $[d] = \{1, 2, \dots, d\}$. We view $\{0, 1\}^d$ in one of two ways, either as a sequence $\epsilon = \epsilon_1 \dots \epsilon_d$ of 0's and 1's, or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\mathbf{0} = (0, 0, \dots, 0) \in \{0, 1\}^d$ is the same as $\emptyset \subset [d]$.

Let $V_d = \{0, 1\}^d = 2^{[d]}$ and $V_d^* = V_d \setminus \{\mathbf{0}\} = V_d \setminus \{\emptyset\}$. If $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$\mathbf{n} \cdot \epsilon = \sum_{i=1}^d n_i \epsilon_i.$$

If we consider ϵ as $\epsilon \subset [d]$, then $\mathbf{n} \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

2.2.2. We denote X^{2^d} by $X^{[d]}$. A point $\mathbf{x} \in X^{[d]}$ can be written in one of two equivalent ways, depending on the context:

$$\mathbf{x} = (x_\epsilon : \epsilon \in \{0, 1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_\emptyset = x_0$ is the first coordinate of \mathbf{x} . For example, points in $X^{[2]}$ are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}).$$

For $x \in X$, we write $x^{[d]} = (x, x, \dots, x) \in X^{[d]}$. The diagonal of $X^{[d]}$ is $\Delta^{[d]} = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$.

A point $\mathbf{x} \in X^{[d]}$ can be decomposed as $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}', \mathbf{x}'' \in X^{[d-1]}$, where $\mathbf{x}' = (x_{\epsilon 0} : \epsilon \in \{0, 1\}^{d-1})$ and $\mathbf{x}'' = (x_{\epsilon 1} : \epsilon \in \{0, 1\}^{d-1})$. We can also isolate the first coordinate, writing $X_*^{[d]} = X^{2^d-1}$ and then writing a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\emptyset, \mathbf{x}_*)$, where $\mathbf{x}_* = (x_\epsilon : \epsilon \neq \emptyset) \in X_*^{[d]}$.

2.3. Dynamical parallelepipeds.

Definition 2.1. Let (X, T) be a t.d.s. and let $d \geq 1$ be an integer. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\mathbf{n} \cdot \epsilon} x = T^{n_1 \epsilon_1 + \dots + n_d \epsilon_d} x : \epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d),$$

where $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $x \in X$. When there is no ambiguity, we write $\mathbf{Q}^{[d]}$ instead of $\mathbf{Q}^{[d]}(X)$. An element of $\mathbf{Q}^{[d]}(X)$ is called a (dynamical) *parallelepiped of dimension d* .

As examples, $\mathbf{Q}^{[2]}$ is the closure in $X^{[2]} = X^4$ of the set

$$\{(x, T^m x, T^n x, T^{n+m} x) : x \in X, m, n \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}$ is the closure in $X^{[3]} = X^8$ of the set

$$\{(x, T^m x, T^n x, T^{m+n} x, T^p x, T^{m+p} x, T^{n+p} x, T^{m+n+p} x) : x \in X, m, n, p \in \mathbb{Z}\}.$$

Definition 2.2. Let $\phi : X \rightarrow Y$ and $d \in \mathbb{N}$. Define $\phi^{[d]} : X^{[d]} \rightarrow Y^{[d]}$ by $(\phi^{[d]} \mathbf{x})_\epsilon = \phi x_\epsilon$ for every $\mathbf{x} \in X^{[d]}$ and every $\epsilon \subset [d]$.

Let (X, T) be a system and $d \geq 1$ be an integer. The *diagonal transformation* of $X^{[d]}$ is the map $T^{[d]}$.

Definition 2.3. *Face transformations* are defined inductively as follows: Let $T^{[0]} = T$, $T_1^{[1]} = \text{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is defined already, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad j \in \{1, 2, \dots, d-1\},$$

$$T_d^{[d]} = \text{id}^{[d-1]} \times T^{[d-1]}.$$

The *face group* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For $\mathcal{G}^{[d]}$ and $\mathcal{F}^{[d]}$, we use similar notations to that used for $X^{[d]}$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0, 1\}^d)$. In particular, $\mathcal{F}^{[d]} = \{S \in \mathcal{G}^{[d]} : S_\emptyset = \text{id}\}$.

For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}(\mathbf{x})}$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$.

It is easy to verify that $\mathbf{Q}^{[d]}$ is the closure in $X^{[d]}$ of

$$\{Sx^{[d]} : S \in \mathcal{F}^{[d]}, x \in X\}.$$

If x is a transitive point of X , then $\mathbf{Q}^{[d]}$ is the closed orbit of $x^{[d]}$ under the group $\mathcal{G}^{[d]}$.

2.4. Nilmanifolds and nilsystems.

2.4.1. *Nilpotent groups.* Let G be a group. For $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h and we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups G_j , $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $k \geq 1$ be an integer. We say that G is *k-step nilpotent* if G_{k+1} is the trivial subgroup.

2.4.2. *Nilmanifolds.* Let G be a k -step nilpotent Lie group and Γ a discrete cocompact subgroup of G . The compact manifold $X = G/\Gamma$ is called a *k-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$ of X . Then (X, T, μ) is called a *basic k-step nilsystem*.

2.4.3. *d-step nilsystem and system of order d.* We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $(X_i, T_i)_{i \in \mathbb{N}}$ are systems with $\text{diam}(X_i) \leq M < \infty$ and $\phi_i : X_{i+1} \rightarrow X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i \in \mathbb{N}} X_i$ given by $\{(x_i)_{i \in \mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\varprojlim \{X_i\}_{i \in \mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x, y) = \sum_{i \in \mathbb{N}} 1/2^i \rho_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

Definition 2.4 (Host-Kra-Maass). [28] A system (X, T) is called a *d-step nilsystem*, if it is an inverse limit of basic d -step nilsystems. A system (X, T) is called a *system of order d*, if it is a minimal d -step nilsystem, equivalently it is an inverse limit of basic d -step minimal nilsystems.

2.5. Families and filters.

2.5.1. *Furstenberg families.* We say that a collection \mathcal{F} of subsets of \mathbb{Z} is a *family* if it is hereditary upward, i.e. $F_1 \subseteq F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family \mathcal{F} is called *proper* if it is neither empty nor the entire power set of \mathbb{Z} , or, equivalently if $\mathbb{Z} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any nonempty collection \mathcal{A} of subsets of \mathbb{Z} generates a family $\mathcal{F}(\mathcal{A}) := \{F \subseteq \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}$.

For a family \mathcal{F} its *dual* is the family $\mathcal{F}^* := \{F \subseteq \mathbb{Z} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}$. It is not hard to see that $\mathcal{F}^* = \{F \subseteq \mathbb{Z} : \mathbb{Z} \setminus F \notin \mathcal{F}\}$, from which we have that if \mathcal{F} is a family then $(\mathcal{F}^*)^* = \mathcal{F}$. For more details, see [1].

2.5.2. *Filter and the Ramsey property.* If a family \mathcal{F} is closed under finite intersections and is proper, then it is called a *filter*.

A family \mathcal{F} has *the Ramsey property* if $A = A_1 \cup A_2 \in \mathcal{F}$ then $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. It is well known that a proper family has the Ramsey property if and only if its dual \mathcal{F}^* is a filter [16].

2.5.3. *Some important families.* A subset S of \mathbb{Z} is *syndetic* if it has a bounded gaps, i.e. there is $N \in \mathbb{N}$ such that $\{i, i+1, \dots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}$. The collection of all syndetic subsets is denoted by \mathcal{F}_s .

Let S be a subset of \mathbb{Z} . The *upper Banach density* and *lower Banach density* of S are

$$BD^*(S) = \limsup_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|}, \text{ and } BD_*(S) = \liminf_{|I| \rightarrow \infty} \frac{|S \cap I|}{|I|},$$

where I ranges over intervals of \mathbb{Z} , while the *upper density* of S is

$$D^*(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap [-n, n]|}{2n+1}.$$

Let $\{b_i\}_{i \in I}$ be a finite or infinite sequence in \mathbb{Z} . One defines

$$FS(\{b_i\}_{i \in I}) = \left\{ \sum_{i \in \alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \right\}.$$

F is an *IP set* if it contains some $FS(\{p_i\}_{i=1}^\infty)$, where $p_i \in \mathbb{Z}$. The collection of all IP sets is denoted by \mathcal{F}_{ip} . A subset of \mathbb{Z} is called an *IP*-set*, if it has non-empty intersection with any IP-set. If I is finite, then one says $FS(\{p_i\}_{i \in I})$ an *finite IP set*. The collection of all sets containing finite IP sets with arbitrarily long lengths is denoted by \mathcal{F}_{fip} .

2.6. **Regionally proximal pairs of order d .** First recall the definition of regionally proximal pairs of order d . Let (X, T) be a t.d.s. and let $d \geq 1$ be an integer. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$, and

$$\rho(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') < \delta \text{ for any nonempty } \epsilon \subset [d].$$

The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}$ (or by $\mathbf{RP}^{[d]}(X)$ in case of ambiguity), which is called *the regionally proximal relation of order d* .

Moreover, let $\mathbf{RP}^{[\infty]} = \bigcap_{d=1}^{\infty} \mathbf{RP}^{[d]}(X)$. The following theorem was proved by Host-Kra-Maass for minimal distal systems [28] and by Shao-Ye [36] for the general minimal systems.

Theorem 2.5. *Let (X, T) be a minimal t.d.s. and $d \in \mathbb{N}$. Then*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $(x, y, y, \dots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ if and only if $(x, x_*^{[d]}, y, x_*^{[d]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.
- (2) $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$.
- (3) $\mathbf{RP}^{[d]}(X)$ is an equivalence relation, and so is $\mathbf{RP}^{[\infty]}$.
- (4) If $\pi : (X, T) \rightarrow (Y, S)$ is a factor map, then $(\pi \times \pi)(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.
- (5) $(X/\mathbf{RP}^{[d]}, T)$ is the maximal nilfactor of (X, T) .

Remark 2.6. In [36], Theorem 2.5 was proved for compact metric spaces. In fact, one can show that Theorem 2.5 holds for compact Hausdorff spaces by repeating the proofs sentence by sentence in [36]. However, we will describe a direct approach in Appendix B. This result will be used in the next section.

3. NIL_d BOHR₀-SETS, POINCARÉ SETS AND $\mathbf{RP}^{[d]}$

In this section using results obtained in [32] we characterize $\mathbf{RP}^{[d]}$ using the families \mathcal{F}_{Poi_d} , \mathcal{F}_{Bir_d} and $\mathcal{F}_{d,0}^*$.

3.1. Nil-Bohr sets. Recall that a subset $A \subseteq \mathbb{Z}$ is a *Nil_d Bohr₀ set* if there exist a d -step nilsystem (X, T) , $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that $\{n \in \mathbb{Z} : T^n x_0 \in U\}$ is contained in A . Denote by $\mathcal{F}_{d,0}$ the family generated by all Nil_d Bohr₀-sets.

For $F_1, F_2 \in \mathcal{F}_{d,0}$, there are d -step nilsystems (X, T) , (Y, S) , $(x, y) \in X \times Y$ and $U \times V$ neighborhood of (x, y) such that $N(x, U) \subset F_1$ and $N(y, V) \subset F_2$. It is clear that $N(x, U) \cap N(y, V) = N((x, y), U \times V) \in \mathcal{F}_{d,0}$. This implies that $F_1 \cap F_2 \in \mathcal{F}_{d,0}$. So we conclude that

Proposition 3.1. *Let $d \in \mathbb{N}$. Then $\mathcal{F}_{d,0}$ is a filter, and $\mathcal{F}_{d,0}^*$ has the Ramsey property.*

3.2. Sets of d -recurrence.

3.2.1. First let us recall some notions. Let $S \subseteq \mathbb{Z}$ and $d \in \mathbb{N}$. We say that S is a set of *d -recurrence* if for all measure preserving systems (X, \mathcal{X}, μ, T) , for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that

$$\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0.$$

We say that S is a set of *d -topological recurrence* if for every minimal t.d.s. (X, T) and for every nonempty open subset U of X , there exists $n \in S$ such that

$$U \cap T^{-n}U \cap \dots \cap T^{-dn}U \neq \emptyset.$$

For $d \in \mathbb{N}$, let \mathcal{F}_{Poi_d} (resp. \mathcal{F}_{Bir_d}) be the family generated by the collection of all sets of d -recurrence (resp. sets of d -topological recurrence).

Remark 3.2. It is known that for all integer $d \geq 2$ there exists a set of $(d - 1)$ -recurrence that is not a set of d -recurrence [13]. This also follows from Theorem 1.8.

Recall that a set $S \subseteq \mathbb{Z}$ is d -intersective if every subset A with positive density contains at least one arithmetic progression of length $d + 1$ and a common difference in S , i.e. there is some $n \in S$ such that

$$A \cap (A - n) \cap (A - 2n) \dots \cap (A - dn) \neq \emptyset.$$

Similarly, one can define topological d -intersective set by replacing the set with positive density with a syndetic set in the above definition.

We now give some equivalence conditions of d -topological recurrence.

Proposition 3.3. *The following statements are equivalent:*

- (1) $S \subset \mathbb{Z}$ is a set of topological d -intersective.
- (2) $S \subset \mathbb{Z}$ is a set of d -topological recurrence.
- (3) For any dynamical system (X, T) there are $x \in X$ and $\{n_i\}_{i=1}^\infty \subset S$ such that

$$\lim_{i \rightarrow +\infty} T^{jn_i} x = x \text{ for each } 1 \leq j \leq d.$$

Proof. The equivalence between (1) and (2) was proved in [13, 15], and the proof of the equivalence between (1) and (3) is similar to the one in Proposition 5.2. \square

3.2.2. A simple and useful fact is:

Proposition 3.4. *For all $d \in \mathbb{N}$, \mathcal{F}_{Poi_d} and \mathcal{F}_{Bir_d} have the Ramsey property.*

Proof. Let $F \in \mathcal{F}_{Poi_d}$ and $F = F_1 \cup F_2$. Assume the contrary that $F_i \notin \mathcal{F}_{Poi_d}$ for $i = 1, 2$. Then there are measure preserving systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and $A_i \in \mathcal{B}_i$ with $\mu_i(A_i) > 0$ such that $\mu_i(A_i \cap T_i^{-n} A_i \cap \dots \cap T_i^{-dn} A_i) = 0$ for $n \in F_i$, where $i = 1, 2$. Set $\mu = \mu_1 \times \mu_2$, $A = A_1 \times A_2$ and $T = T_1 \times T_2$. Then we have

$$\begin{aligned} & \mu(A \cap T^{-n} A \cap \dots \cap T^{-dn} A) \\ &= \mu_1(A_1 \cap T_1^{-n} A_1 \cap \dots \cap T_1^{-dn} A_1) \mu_2(A_2 \cap T_2^{-n} A_2 \cap \dots \cap T_2^{-dn} A_2) = 0 \end{aligned}$$

for each $n \in F = F_1 \cup F_2$, a contradiction. The other case can be shown similarly. \square

3.3. **Nil_d Bohr₀-sets and $\mathbf{RP}^{[d]}$.** To show the following result we need several well known facts from the Ellis enveloping semigroup theory, see [2, 19, 40, 41]. Also we note that the lifting property in Theorem 2.5 is valid when X is compact and Hausdorff (see Appendix B for more details).

Theorem 3.5. *Let (X, T) be a minimal t.d.s.. Then $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{d,0}^*$ for each neighborhood U of y .*

Proof. First assume that $N(x, U) \in \mathcal{F}_{d,0}^*$ for each neighborhood U of y . Let (X_d, S) be the maximal d -step nilfactor of (X, T) (see Theorem 2.5) and $\pi : X \rightarrow X_d$ be the projection. Then for any neighborhood V of $\pi(x)$, we have $N(x, U) \cap N(\pi(x), V) \neq \emptyset$ since $N(x, U) \in \mathcal{F}_{d,0}^*$. This means that there is a sequence $\{n_i\}$ such that

$$(T \times S)^{n_i}(x, \pi(x)) \rightarrow (y, \pi(x)), \quad i \rightarrow \infty.$$

Thus, we have

$$\pi(y) = \pi(\lim_i T^{n_i} x) = \lim_i S^{n_i} \pi(x) = \pi(x),$$

i.e. $(x, y) \in \mathbf{RP}^{[d]}$.

Now assume that $(x, y) \in \mathbf{RP}^{[d]}$ and U is a neighborhood of y . We need to show that if (Z, R) is a d -step nilsystem, $z_0 \in Z$ and V is a neighborhood of z_0 then $N(x, U) \cap N(z_0, V) \neq \emptyset$.

Let

$$W = \prod_{z \in Z} Z \quad (\text{i.e. } W = Z^Z) \text{ and } R^Z : W \rightarrow W$$

with $(R^Z \omega)(z) = R(\omega(z))$ for any $z \in Z$, where $\omega = (\omega(z))_{z \in Z} \in W$. Note that in general (W, R^Z) is not a metrizable but a compact Hausdorff system. Since (Z, R) is a d -step nilsystem, (Z, R) is distal. Hence (W, R^Z) is also distal.

Choose $\omega^* \in W$ with $\omega^*(z) = z$ for $z \in Z$, and let $Z_\infty = \text{cl}(\text{orb}(\omega^*, R^Z))$. Then (Z_∞, R^Z) is a minimal subsystem of (W, R^Z) since (W, R^Z) is distal. For $\omega \in Z_\infty$, there exists $p \in E(Z, R)$ such that $\omega(z) = p(\omega^*(z)) = p(z)$ for $z \in Z$. Since (Z, R) is a minimal distal system, the Ellis semigroup $E(Z, R)$ is a group (Appendix B). Particularly, $p : Z \rightarrow Z$ is a surjective map. Thus

$$\{\omega(z) : z \in Z\} = \{p(z) : z \in Z\} = Z.$$

Hence there exists $z_\omega \in Z$ such that $\omega(z_\omega) = z_0$.

Take a minimal subsystem $(A, T \times R^Z)$ of the product system $(X \times Z_\infty, T \times R^Z)$. Let $\pi_X : A \rightarrow X$ be the natural coordinate projection. Then $\pi_X : (A, T \times R^Z) \rightarrow (X, T)$ is a factor map between two minimal systems. Since $(x, y) \in \mathbf{RP}^{[d]}(X, T)$, by Theorem 2.5 there exist $\omega^1, \omega^2 \in W$ such that $((x, \omega^1), (y, \omega^2)) \in \mathbf{RP}^{[d]}(A, T \times R^Z)$.

For ω^1 , there exists $z_1 \in Z$ such that $\omega^1(z_1) = z_0$ by the above discussion. Let $\pi : A \rightarrow X \times Z$ with $\pi(u, \omega) = (u, \omega(z_1))$ for $(u, \omega) \in A$, $u \in X$, $\omega \in W$. Let $B = \pi(A)$. Then $(B, T \times R)$ is a minimal subsystem of $(X \times Z, T \times R)$, and $\pi : (A, T \times R^Z) \rightarrow (B, T \times R)$ is a factor map between two minimal systems. Clearly $\pi(x, \omega^1) = (x, z_0)$, $\pi(y, \omega^2) = (y, z_2)$ for some $z_2 \in Z$, and

$$((x, z_0), (y, z_2)) = \pi \times \pi((x, \omega^1), (y, \omega^2)) \in \mathbf{RP}^{[d]}(B, T \times R).$$

Moreover, we consider the projection π_Z of B onto Z . Then $\pi_Z : (B, T \times R) \rightarrow (Z, R)$ is a factor map and so $(z_0, z_2) = \pi_Z \times \pi_Z((x, z_0), (y, z_2)) \in \mathbf{RP}^{[d]}(Z, R)$. Since (Z, R) is a d -step nilsystem, $z_0 = z_2$. Thus $((x, z_0), (y, z_0)) \in \mathbf{RP}^{[d]}(B, T \times R)$. Particularly, $N(x, U) \cap N(z_0, V) = N((x, z_0), U \times V)$ is a syndetic set since $(B, T \times R)$ is minimal. This completes the proof of theorem. \square

Remark 3.6. From the proof of Theorem 3.5, we have the following result: Let (X, T) be a minimal system and $(x, y) \in \mathbf{RP}^{[d]}$. Then $N(x, U) \cap F$ is a syndetic set for each $F \in \mathcal{F}_{d,0}$ and each neighborhood U of y .

3.4. Sets of d -recurrence and nilsequences. It is known that d -recurrence sets are “almost” d -step nilsequences [32]. This result stated in Theorem 3.9 follows from Propositions 3.7 and 3.8 by a discussion in [32].

Proposition 3.7. [5, Theorem 1.9] *Let (X, \mathcal{X}, μ, T) be an ergodic measure preserving system, let $f \in L^\infty(\mu)$ and let $d \geq 1$ be an integer. The sequence $\{I_f(d, n)\}$ is the sum of a sequence tending to zero in uniform density and a d -step nilsequence, where*

$$(3.1) \quad I_f(d, n) = \int f(x)f(T^n x) \dots f(T^{dn} x) d\mu(x).$$

Especially, for any $A \in \mathcal{X}$

$$\{I_{1_A}(d, n)\} = \{\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A)\} = F_d + N,$$

where F_d is a d -step nilsequence and N tending to zero in uniform density.

Proposition 3.8. [17] or [6, Theorem 6.15] *Let (X, \mathcal{X}, μ, T) be an ergodic measure preserving system and $d \in \mathbb{N}$. Then for $A \in \mathcal{X}$ with $\mu(A) > 0$ there is $c > 0$ such that*

$$\{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > c\}$$

is an IP^ -set.*

Theorem 3.9. *Let (X, \mathcal{X}, μ, T) be an ergodic measure preserving system and $d \in \mathbb{N}$. Then for all $A \in \mathcal{X}$ with $\mu(A) > 0$ the set*

$$I = \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0\}$$

is an “almost” Nil_d Bohr $_0$ -set, i.e. there is some subset M with $BD^(M) = 0$ such that $I\Delta M$ is a Nil_d Bohr $_0$ -set.*

As an immediate consequence, one has

Corollary 3.10. *Let (X, T) be a minimal t.d.s. and $d \in \mathbb{N}$. If $(x, y) \in \mathbf{RP}^{[d]}$, then $N(x, U) \in \mathcal{F}_{Poi_d}$ and $N(x, U) \in \mathcal{F}_{Bir_d}$ for each neighborhood U of y .*

Proof. Let U be a neighborhood of y . We have shown in Theorem 3.5 that $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{d,0}^*$. This means that $N(x, U) \cap B \neq \emptyset$ for each $B \in \mathcal{F}_{d,0}$.

Now let (X, \mathcal{X}, μ, T) be an ergodic measure preserving system and $A \in \mathcal{X}$ with $\mu(A) > 0$. Set

$$F = \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0\}.$$

By Theorem 3.9 there is some subset M with $BD^*(M) = 0$ such that $B = F\Delta M$ is a Nil_d Bohr $_0$ -set. Hence we have $N(x, U) \cap (F\Delta M)$ is syndetic by Remark 3.6. Thus we conclude that there is $n \neq 0$ with $n \in N(x, U) \cap F$ since $BD^*(M) = 0$. By the definition, $N(x, U) \in \mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d}$. The proof is completed. \square

3.5. A result concerning Nil_d Bohr₀-sets. To show the converse of Corollary 3.10, we need the following result.

Theorem 3.11. [32] *Let $d \in \mathbb{N}$. Then*

$$\mathcal{F}_{\text{Poi}_d} \subset \mathcal{F}_{\text{Bir}_d} \subset \mathcal{F}_{d,0}^*.$$

3.6. Recurrence sets and $\mathbf{RP}^{[d]}$. Now we can sum up the main results of this section as follows:

Theorem 3.12. *Let (X, T) be a minimal t.d.s.. Then the following statements are equivalent:*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$.
- (2) $N(x, U) \in \mathcal{F}_{\text{Poi}_d}$ for each neighborhood U of y .
- (3) $N(x, U) \in \mathcal{F}_{\text{Bir}_d}$ for each neighborhood U of y .
- (4) $N(x, U) \in \mathcal{F}_{d,0}^*$ for each neighborhood U of y .

Proof. By Corollary 3.10 one has that (1) \Rightarrow (2). It follows from Theorem 3.11 that (2) \Rightarrow (3) \Rightarrow (4). By Theorem 3.5, one has that (4) \Rightarrow (1) and completes the proof. \square

4. SG_d -SETS AND $\mathbf{RP}^{[d]}$

In this section we will describe $\mathbf{RP}^{[d]}$ using the SG_d -sets introduced by Host and Kra in [27]. First we recall some definitions.

4.1. Sets $SG_d(P)$.

Definition 4.1. Let $d \geq 0$ be an integer and let $P = \{p_i\}_i$ be a (finite or infinite) sequence in \mathbb{Z} . The *set of sums with gaps of length less than d* of P is the set $SG_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \dots + \epsilon_n p_n$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than d .

A subset $A \subseteq \mathbb{Z}$ is an SG_d^* -set if $A \cap SG_d(P) \neq \emptyset$ for every infinite sequence P in \mathbb{Z} .

Note that in this definition, P is a sequence and not a subset of \mathbb{Z} . For example, if $P = \{p_1, p_2, \dots\}$, then $SG_1(P)$ is the set of all sums $p_m + p_{m+1} + \dots + p_n$ of consecutive elements of P , and thus it coincides with the set $\Delta(S)$ where $S = \{p_1, p_1 + p_2, p_1 + p_2 + p_3, \dots\}$. Therefore SG_1^* -sets are the same as Δ^* -sets.

For a sequence P , $SG_2(P)$ consists of all sums of the form

$$\sum_{i=m_0}^{m_1} p_i + \sum_{i=m_1+2}^{m_2} p_i + \dots + \sum_{i=m_{k-1}+2}^{m_k} p_i + \sum_{i=m_k+2}^{m_{k+1}} p_i$$

where $k \in \mathbb{N}$ and m_0, m_1, \dots, m_{k+1} are positive integers satisfying $m_{i+1} \geq m_i + 2$ for $i = 0, 1, \dots, k$.

Denote by SG_d the collection of all sets $SG_d(P)$ with P infinite, and \mathcal{F}_{SG_d} the family generated by SG_d for each $d \in \mathbb{N}$. Moreover, let \mathcal{F}_{fSG_d} be the family containing arbitrarily long $SG_d(P)$ sets with P finite. That is, $A \in \mathcal{F}_{fSG_d}$ if and only if there are finite sets P^i with $|P^i| \rightarrow \infty$ such that $\bigcup_{i=1}^{\infty} SG_d(P^i) \subset A$. It is clear that

$$\mathcal{F}_{SG_1} \supset \mathcal{F}_{SG_2} \supset \dots \supset \mathcal{F}_{SG_{\infty}} =: \bigcap_{i=1}^{\infty} \mathcal{F}_{SG_i},$$

and

$$\mathcal{F}_{fSG_1} \supset \mathcal{F}_{fSG_2} \supset \dots \supset \mathcal{F}_{fSG_{\infty}} =: \bigcap_{i=1}^{\infty} \mathcal{F}_{fSG_i}.$$

We now show

Proposition 4.2. *The following statements hold:*

- (1) $\mathcal{F}_{SG_{\infty}} = \{A : \exists P^i \text{ infinite for each } i \in \mathbb{N} \text{ such that } A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)\}$.
- (2) $\mathcal{F}_{fSG_{\infty}} = \mathcal{F}_{fip}$.

Proof. (1). Assume that $A \in \mathcal{F}_{SG_{\infty}}$. Then $A \in \bigcap_{i=1}^{\infty} \mathcal{F}_{SG_i}$ and hence $A \in \mathcal{F}_{SG_i}$ for each $i \in \mathbb{N}$. Thus for each $i \in \mathbb{N}$ there is P^i infinite such that $A \supset SG_i(P^i)$ which implies that $A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)$.

Now let $B = \bigcup_{i=1}^{\infty} SG_i(P^i)$, where P^i infinite for each $i \in \mathbb{N}$. It is clear that $B \subset \mathcal{F}_{SG_i}$ for each i and thus, $B \in \mathcal{F}_{SG_{\infty}}$. Since $\mathcal{F}_{SG_{\infty}}$ is a family, we conclude that $\{A : \exists P^i \text{ infinite for each } i \in \mathbb{N} \text{ such that } A \supset \bigcup_{i=1}^{\infty} SG_i(P^i)\} \subset \mathcal{F}_{SG_{\infty}}$.

(2) It is clear that $\mathcal{F}_{fSG_{\infty}} \subset \mathcal{F}_{fip}$. Let $A \in \mathcal{F}_{fip}$ and without loss of generality assume that $A = \bigcup_{i=1}^{\infty} FS(P^i)$ with $P^i = \{p_1^i, \dots, p_i^i\}$ and $|P^i| \rightarrow \infty$.

Put $A_d = \bigcup_{i=1}^{\infty} SG_d(P^i) \subset A$ for $d \in \mathbb{N}$. Then $A_d \in \mathcal{F}_{fSG_d}$ which implies that $A \in \mathcal{F}_{fSG_d}$ for each $d \geq 1$ and hence $A \in \mathcal{F}_{fSG_{\infty}}$. That is, $\mathcal{F}_{fip} \subset \mathcal{F}_{fSG_{\infty}}$. \square

4.2. SG_d -sets and $\mathbf{RP}^{[d]}$. The following theorem is the main result of this section.

Theorem 4.3. *Let (X, T) be a minimal t.d.s.. Then for any $d \in \mathbb{N}$, $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{SG_d}$ for each neighborhood U of y . The same holds when $d = \infty$.*

Proof. It is clear that if $N(x, U) \in \mathcal{F}_{SG_d}$ for each neighborhood U of y , then it contains some $FS(\{n_i\}_{i=1}^{d+1})$ for each neighborhood U of y which implies that $(x, y) \in \mathbf{RP}^{[d]}$ by Theorem 2.5.

Now assume that $(x, y) \in \mathbf{RP}^{[d]}$ for $d \geq 1$. Let for $i \geq 2$

$$A_i =: \{0, 1\}^i \setminus \{(0, \dots, 0, 0), (0, \dots, 0, 1)\}$$

The case when $d = 1$ was proved by Veech [39] and our method is also valid for this case. To make the idea of the proof clearer, we first show the case when $d = 2$ and the general case follows by the same idea.

I. The case $d = 2$.

Assume that $(x, y) \in \mathbf{RP}^{[2]}$. Then by Theorem 2.5 (1) and (2) for each neighborhood $V \times U$ of (x, y) , there are $n_1, n_2, n_3 \in \mathbb{N}$ such that

$$T^{\epsilon_1 n_1 + \epsilon_2 n_2 + \epsilon_3 n_3} x \in V \text{ and } T^{n_3} x \in U,$$

where $(\epsilon_1, \epsilon_2, \epsilon_3) \in A_3$. For a given U , let $\eta > 0$ with $B(y, \eta) \subset U$, and take $\eta_i > 0$ with $\sum_{i=1}^{\infty} \eta_i < \eta$, where $B(y, \eta) = \{x : \rho(x, y) < \eta\}$.

Choose $n_1^1, n_2^1, n_3^1 \in \mathbb{N}$ such that

$$\rho(T^{n_3^1} x, y) < \eta_1 \text{ and } \rho(T^r x, x) < \eta_1,$$

where $r \in E_1$ with

$$E_1 = \{\epsilon_1 n_1^1 + \epsilon_2 n_2^1 + \epsilon_3 n_3^1 : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_1 = FS(\{n_1^1, n_2^1, n_3^1\}).$$

Choose $n_1^2, n_2^2, n_3^2 \in \mathbb{N}$ such that

$$\rho(T^{n_3^2} x, y) < \eta_2 \text{ and } \max_{s \in S_1} \rho(T^{s+r} x, T^s x) < \eta_2$$

for each $r \in E_2$ with

$$E_2 = \{\epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \epsilon_3 n_3^2 : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_2 = FS(\{n_i^j : j = 1, 2, i = 1, 2, 3\}).$$

Generally when $n_1^i, n_2^i, n_3^i, E_i, S_i$ are defined for $1 \leq i \leq k$ choose $n_1^{k+1}, n_2^{k+1}, n_3^{k+1} \in \mathbb{N}$ such that

$$(4.1) \quad \rho(T^{n_3^{k+1}} x, y) < \eta_{k+1} \text{ and } \max_{s \in S_k} \rho(T^{s+r} x, T^s x) < \eta_{k+1}.$$

for each $r \in E_{k+1}$, where

$$E_{k+1} = \{\epsilon_1 n_1^{k+1} + \epsilon_2 n_2^{k+1} + \epsilon_3 n_3^{k+1} : (\epsilon_1, \epsilon_2, \epsilon_3) \in A_3\}.$$

Let

$$S_{k+1} = FS(\{n_i^j : i = 1, 2, 3, 1 \leq j \leq k+1\}).$$

Now we define a subsequence $P = \{P_k\}$ such that

$$P_1 = n_3^1 + n_1^2 + n_1^3, P_2 = n_3^2 + n_2^3 + n_2^4, P_3 = n_3^3 + n_1^4 + n_1^5, P_4 = n_3^4 + n_2^5 + n_2^6, \dots$$

That is,

$$P_k = n_3^k + n_{k \pmod{2}}^{k+1} + n_{k \pmod{2}}^{k+2},$$

where we assume $2m \pmod{2} = 2$ for $m \in \mathbb{N}$. We claim that $N(x, U) \supset SG_2(P)$.

Let $n \in SG_2(P)$ then $n = \sum_{j=1}^k P_{i_j}$, where $1 \leq i_{j+1} - i_j \leq 2$ for $1 \leq j \leq k-1$. By induction for k , it is not hard to show that n can be written as

$$n = a_1 + a_2 + \dots + a_{i_k - i_1 + 3}$$

such that $a_1 = n_3^{i_1}$, $a_j \in E_{j+i_1-1}$ for $j = 2, 3, \dots, i_k - i_1 + 1$ and $a_{i_k-i_1+2} \in \{n_1^{i_k+1}, n_2^{i_k+1}, n_1^{i_k+1} + n_2^{i_k+1}\}$, $a_{i_k-i_1+3} = n_{i_k}^{i_k+2} \pmod{2}$. In other words, n can be written as $n = a_1 + a_2 + \dots + a_{i_k-i_1+3}$ with $a_1 = n_3^{i_1}$ and $a_j \in E_{i_1+j-1}$ for $2 \leq j \leq i_k - i_1 + 3$.

Note that $\sum_{\ell=1}^j a_\ell \in S_{i_1+\ell-1}$ and $a_{j+1} \in E_{i_1+j}$ for $1 \leq j \leq i_k - i_1 + 2$. Thus by (4.1) we have

$$\rho(T^{\sum_{i=1}^j a_i} x, T^{\sum_{i=1}^{j+1} a_i} x) < \eta_{j+i_1}$$

for $1 \leq j \leq i_k - i_1 + 2$. This implies that

$$\begin{aligned} \rho(T^n x, y) &\leq \rho(T^{\sum_{j=1}^{i_k-i_1+3} a_j} x, T^{\sum_{j=1}^{i_k-i_1+2} a_j} x) + \dots + \rho(T^{n_3^{i_1}+a_2} x, T^{n_3^{i_1}} x) + \rho(T^{n_3^{i_1}} x, y) \\ &< \sum_{j=0}^{i_k-i_1+2} \eta_{j+i_1} < \eta. \end{aligned}$$

That is, $n \in N(x, U)$ and hence $N(x, U) \supset SG_2(P)$.

II. The general case.

Generally assume that $(x, y) \in \mathbf{RP}^{[d]}$ with $d \geq 2$. Then by Theorem 2.5 (1) and (2) for each neighborhood $V \times U$ of (x, y) , there are $n_1, n_2, \dots, n_{d+1} \in \mathbb{N}$ such that

$$T^{\epsilon_1 n_1 + \epsilon_2 n_2 + \dots + \epsilon_{d+1} n_{d+1}} x \in V \text{ and } T^{n_{d+1}} x \in U,$$

where $(\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}$. For a given U , let $\eta > 0$ with $B(y, \eta) \subset U$, and take $\eta_i > 0$ with $\sum_{i=1}^{\infty} \eta_i < \eta$.

Choose $n_1^1, n_2^1, \dots, n_{d+1}^1 \in \mathbb{N}$ such that $\rho(T^{n_{d+1}^1} x, y) < \eta_1$ and $\rho(T^r x, x) < \eta_1$ where $r \in E_1$ with

$$E_1 = \{\epsilon_1 n_1^1 + \epsilon_2 n_2^1 + \dots + \epsilon_{d+1} n_{d+1}^1 : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_1 = FS(\{n_1^1, \dots, n_{d+1}^1\}).$$

Choose $n_1^2, n_2^2, \dots, n_{d+1}^2$ such that

$$\rho(T^{n_{d+1}^2} x, y) < \eta_2, \text{ and } \max_{s \in S_1} \rho(T^{s+r} x, T^s x) < \eta_2$$

for each $r \in E_2$ with

$$E_2 = \{\epsilon_1 n_1^2 + \epsilon_2 n_2^2 + \dots + \epsilon_{d+1} n_{d+1}^2 : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_2 = FS(\{n_1^1, \dots, n_{d+1}^1, n_1^2, \dots, n_{d+1}^2\}).$$

Generally when $n_1^i, \dots, n_{d+1}^i, E_i, S_i$ are defined for $1 \leq i \leq k$ choose $n_1^{k+1}, \dots, n_{d+1}^{k+1} \in \mathbb{N}$ such that

$$(4.2) \quad \rho(T^{n_{d+1}^{k+1}} x, y) < \eta_{k+1}, \text{ and } \max_{s \in S_k} \rho(T^{s+r} x, T^s x) < \eta_{k+1}.$$

for each $r \in E_{k+1}$, where

$$E_{k+1} = \{\epsilon_1 n_1^{k+1} + \epsilon_2 n_2^{k+1} + \dots + \epsilon_{d+1} n_{d+1}^{k+1} : (\epsilon_1, \epsilon_2, \dots, \epsilon_{d+1}) \in A_{d+1}\}.$$

Let

$$S_{k+1} = FS(\{n_i^j : i = 1, \dots, d+1, 1 \leq j \leq k+1\}).$$

Now we define a subsequence $P = \{P_k\}$ such that

$$\begin{aligned} P_1 &= n_{d+1}^1 + n_1^2 + \dots + n_1^{d+1}, P_2 = n_{d+1}^2 + n_2^3 + \dots + n_2^{d+2}, \dots, \\ P_d &= n_{d+1}^d + n_d^{d+1} + \dots + n_d^{2d}, \\ P_{d+1} &= n_{d+1}^{d+1} + n_1^{d+2} + \dots + n_1^{2d+1}, P_{d+2} = n_{d+1}^{d+2} + n_2^{d+3} + \dots + n_2^{2d+2}, \dots, \\ P_{2d} &= n_{d+1}^{2d} + n_d^{2d+1} + \dots + n_d^{3d}, \dots \end{aligned}$$

That is,

$$P_k = n_{d+1}^k + n_{k \pmod{d}}^{k+1} + \dots + n_{k \pmod{d}}^{k+d},$$

where we assume $dm \pmod{d} = d$ for $m \in \mathbb{N}$.

We claim that $N(x, U) \supset SG_d(P)$. Let $n \in SG_d(P)$ then $n = \sum_{j=1}^k P_{i_j}$, where $1 \leq i_{j+1} - i_j \leq d$ for $1 \leq j \leq k-1$. By induction for k , it is not hard to show that n can be written as

$$n = a_1 + a_2 + \dots + a_{i_k - i_1 + d + 1}$$

such that $a_1 = n_{d+1}^{i_1}$, $a_j \in E_{j+i_1-1}$ for $j = 2, 3, \dots, i_k - i_1 + 1$ and

$$a_{i_k - i_1 + 1 + r} \in FS\left(\left\{n_\ell^{i_k + r} : \ell \in \{1, 2, \dots, d\} \setminus \bigcup_{j=1}^{r-1} \{i_k + j \pmod{d}\}\right\}\right)$$

for $1 \leq r \leq d$. In other words, n can be written as $n = a_1 + a_2 + \dots + a_{i_k - i_1 + d + 1}$ with $a_1 = n_{d+1}^{i_1}$ and $a_j \in E_{i_1 + j - 1}$ for $2 \leq j \leq i_k - i_1 + d + 1$.

Note that $\sum_{\ell=1}^j a_\ell \in S_{i_1 + \ell - 1}$ and $a_{j+1} \in E_{i_1 + j}$ for $1 \leq j \leq i_k - i_1 + d$. Thus by (4.2) we have

$$\rho(T^{\sum_{i=1}^j a_i} x, T^{\sum_{i=1}^{j+1} a_i} x) < \eta_{i_1 + j}$$

for $1 \leq j \leq i_k - i_1 + d$. This implies that

$$\begin{aligned} \rho(T^n x, y) &\leq \rho(T^{\sum_{j=1}^{i_k - i_1 + d + 1} a_j} x, T^{\sum_{j=1}^{i_k - i_1 + d} a_j} x) + \dots + \rho(T^{n_{d+1}^{i_1}} x, y) \\ &< \sum_{j=0}^{i_k - i_1 + d} \eta_{j+i_1} < \eta. \end{aligned}$$

That is, $n \in N(x, U)$ and hence $N(x, U) \supset SG_d(P)$ which implies that $N(x, U) \in \mathcal{F}_{SG_d}$. The proof is completed. \square

5. CUBIC VERSION OF MULTIPLE RECURRENCE SETS AND $\mathbf{RP}^{[d]}$

Cubic version of multiple ergodic averages was studied in [25], and also was proved very useful in some other questions [26, 27, 28].

In this section we will discuss the question how to describe $\mathbf{RP}^{[d]}$ using cubic version of multiple recurrence sets. Since by Theorem 2.5 one can use dynamical parallelepipeds to characterize $\mathbf{RP}^{[d]}$, it seems natural to describe $\mathbf{RP}^{[d]}$ using the cubic version of multiple recurrence sets.

5.1. Cubic version of multiple Birkhoff recurrence sets. First we give definitions for the cubic version of multiple recurrence sets. We leave the equivalent statements in viewpoint of intersective sets in Appendix C.

5.1.1. Birkhoff recurrence sets. First we recall the classical definition. Let $P \subset \mathbb{Z}$. P is called a *Birkhoff recurrence set* (or a *set of topological recurrence*) if whenever (X, T) is a minimal t.d.s. and $U \subseteq X$ a nonempty open set, then $P \cap N(U, U) \neq \emptyset$. Let \mathcal{F}_{Bir} denote the collection of Birkhoff recurrence subsets of \mathbb{Z} . An alternative definition is that for any t.d.s. (X, T) there are $\{n_i\} \subset P$ and $x \in X$ such that $T^{n_i}x \rightarrow x$. Now we generalize the above definition to the higher dimension.

Definition 5.1. Let $P \subset \mathbb{Z}$ and $d \in \mathbb{N}$. P is called a *Birkhoff recurrence set of order d* (or a *set of topological recurrence of order d*) if whenever (X, T) is a t.d.s. there are $x \in X$ and $\{n_i^j\}_{j=1}^d \subset P$, $i \in \mathbb{N}$, such that $FS(\{n_i^j\}_{j=1}^d) \subset P$, $i \in \mathbb{N}$ and for each given $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$, $T^{m_i}x \rightarrow x$, where $m_i = \epsilon_1 n_i^1 + \dots + \epsilon_d n_i^d$, $i \in \mathbb{N}$. A subset F of \mathbb{Z} is a *Birkhoff recurrence set of order ∞* if it is a Birkhoff recurrence set of order d for any $d \geq 1$.

For example, when $d = 2$ this means that there are sequence $\{n_i\}, \{m_i\} \subset P$ and $x \in X$ such that $\{n_i + m_i\} \subset P$ and $T^{n_i}x \rightarrow x, T^{m_i}x \rightarrow x, T^{n_i+m_i}x \rightarrow x$.

Similarly we can define (topologically) intersective of order d and intersective of order d (see Appendix C). We have

Proposition 5.2. *Let $d \in \mathbb{N}$. The following statements are equivalent:*

- (1) P is a Birkhoff recurrence set of order d .
- (2) Whenever (X, T) is a minimal t.d.s. and $U \subseteq X$ a nonempty open set, then there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ such that

$$U \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \right) \neq \emptyset.$$

- (3) P is (topologically) intersective of order d .

Proof. (1) \Rightarrow (2). Assume first that P is a Birkhoff recurrence set of order d . Let (X, T) be a minimal t.d.s. and $U \subseteq X$ a nonempty open set. Then there are $x \in X$ and $\{n_i^j\}_{j=1}^d \subset P$, $i \in \mathbb{N}$, such that for each given $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0, 1\}^d$, $T^{m_i}x \rightarrow x$, where $m_i = \epsilon_1 n_i^1 + \dots + \epsilon_d n_i^d$, $i \in \mathbb{N}$. Since (X, T) is minimal, there is some $k \in \mathbb{Z}$ such that $x \in V = T^{-k}U$. When i_0 is larger enough, we have $V \cap \left(\bigcap_{n \in FS(\{n_{i_0}^j\}_{j=1}^d)} T^{-n}V \right) \neq \emptyset$, which implies that $U \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \right) \neq \emptyset$ by putting $n_i = n_{i_0}^i$.

(2) \Rightarrow (1). Now assume that whenever (Y, S) is a minimal t.d.s. and $V \subseteq Y$ a nonempty open set, then there are $FS(\{n_i\}_{i=1}^d) \subset P$ such that

$$V \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}V \right) \neq \emptyset.$$

Let (X, T) be a t.d.s., and without loss of generality we assume that (X, T) is minimal, since each t.d.s. contains a minimal subsystem. Define for each $j \in \mathbb{N}$

$$W_j = \{x \in X : \exists FS(\{n_i\}_{i=1}^d) \subset P \text{ with } d(T^{m_\epsilon} x, x) < \frac{1}{j} \text{ for each } \epsilon \in \{0, 1\}^d\},$$

where $m_\epsilon = \epsilon_1 n_1^1 + \dots + \epsilon_d n_d^d$. Then it is easy to verify that W_j is non-empty, open and dense. Then any $x \in \bigcap_{j=1}^{\infty} W_j$ is the point we look for.

(1) \Leftrightarrow (3). See Appendix C. \square

Remark 5.3. From the above proof, one can see that for a minimal t.d.s. the set of recurrent point in the Definition 5.1 is residual.

5.1.2. *Some properties of Birkhoff sequences of order d .* The family generated by the collection of all Birkhoff recurrence sets of order d is denoted by \mathcal{F}_{B_d} . We have

$$\mathcal{F}_{B_1} \supset \mathcal{F}_{B_2} \supset \dots \supset \mathcal{F}_{B_d} \supset \dots \supset \mathcal{F}_{B_\infty} =: \bigcap_{d=1}^{\infty} \mathcal{F}_{B_d}.$$

We will show later (after Proposition 5.10) that

Proposition 5.4. $\mathcal{F}_{B_\infty} = \mathcal{F}_{\text{fip}}$.

5.2. **Birkhoff recurrence sets and $\mathbf{RP}^{[d]}$.** We have the following theorem

Theorem 5.5. *Let (X, T) be a minimal t.d.s.. Then for any $d \in \mathbb{N} \cup \{\infty\}$, $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{B_d}$ for each neighborhood U of y .*

Proof. We first show the case when $d \in \mathbb{N}$. (\Leftarrow) Let $d \in \mathbb{N}$ and assume $N(x, U) \in \mathcal{F}_{B_d}$. Then there are $FS(\{n_i\}_{i=1}^d) \subset N(x, U)$ such that $U \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \neq \emptyset$. This means that there is $y' \in U$ such that $T^n y' \in U$ for any $n \in FS(\{n_i\}_{i=1}^d)$. Since $T^n x \in U$ for any $n \in FS(\{n_i\}_{i=1}^d)$, we conclude that $(x, y) \in \mathbf{RP}^{[d]}$ by the definition.

(\Rightarrow) Assume that $(x, y) \in \mathbf{RP}^{[d]}$ and U is a neighborhood of y . Let (Z, R) be a minimal t.d.s., V be a non-empty open subset of Z and $\Lambda \subset X \times Z$ be a minimal subsystem. Let $\pi : \Lambda \rightarrow X$ be the projection. Since $(x, y) \in \mathbf{RP}^{[d]}$ there are $z_1, z_2 \in Z$ such that $((x, z_1), (y, z_2)) \in \mathbf{RP}^{[d]}(\Lambda, T \times R)$ by Theorem 2.5. Let $m \in \mathbb{N}$ such that $T^{-m}V$ be a neighborhood of z_2 . Then $U \times T^{-m}V$ is a neighborhood of (y, z_2) . By Theorem 2.5, there are n_1, \dots, n_{d+1} such that

$$N((x, z_1), U \times T^{-m}V) \supset FS(\{n_i\}_{i=1}^{d+1}).$$

This implies that $\bigcap_{n \in FS(\{n_i\}_{i=1}^{d+1})} T^{-n-m}V \neq \emptyset$. Thus, $V \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}V \neq \emptyset$, i.e. $N(x, U) \in \mathcal{F}_{B_d}$.

The case $d = \infty$ is followed from the result for $d \in \mathbb{N}$ and the definitions. \square

5.3. **Cubic version of multiple Poincaré recurrence sets.**

5.3.1. *Poincaré recurrence sets.* Now we give the cubic version of multiple Poincaré recurrence sets.

Definition 5.6. For $d \in \mathbb{N}$, a subset F of \mathbb{Z} is a *Poincaré sequence of order d* if for each (X, \mathcal{B}, μ, T) and $A \in \mathcal{B}$ with positive measure there are $n_1, \dots, n_d \in \mathbb{Z}$ such that $FS(\{n_i\}_{i=1}^d) \subset F$ and

$$\mu(A \cap (\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A)) > 0.$$

A subset F of \mathbb{Z} is a *Poincaré sequence of order ∞* if it is a Poincaré sequence of order d for any $d \geq 1$.

Remark 5.7. We remark that F is a Poincaré sequence of order 1 iff it is a Poincaré sequence. Moreover, a Poincaré sequence of order 1 does not imply that it is a Poincaré sequence of order 2. For example, $\{n^k : n \in \mathbb{N}\}$ ($k \geq 3$) is a Poincaré sequence [16], it is not a Poincaré sequence of order 2 by the famous Fermat Last Theorem.

5.3.2. *Some properties of Poincaré sequences of order d .* Let for $d \in \mathbb{N} \cup \{\infty\}$, \mathcal{F}_{P_d} be the family generated by the collection of all Poincaré sequences of order d . Thus

$$\mathcal{F}_{P_1} = \mathcal{F}_{P_{oi}} \supset \mathcal{F}_{P_2} \supset \dots \supset \mathcal{F}_{P_d} \supset \dots \supset \mathcal{F}_{P_\infty} =: \bigcap_{d=1}^{\infty} \mathcal{F}_{P_d}.$$

We want to show that $\mathcal{F}_{P_\infty} = \mathcal{F}_{fip}$. It is clear that $\mathcal{F}_{P_\infty} \subset \mathcal{F}_{fip}$. To show $\mathcal{F}_{P_d} \supset \mathcal{F}_{fip}$, we need the following proposition, for a proof see [18] or [31].

Proposition 5.8. *Let (X, \mathcal{B}, μ) be a probability space, and $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets with $\mu(E_i) \geq a > 0$ for some constant a and any $i \in \mathbb{N}$. Then for any $k \geq 1$ and $\epsilon > 0$ there is $N = N(a, k, \epsilon)$ such that for any tuple $\{s_1 < s_2 < \dots < s_n\}$ with $n \geq N$ there exist $1 \leq t_1 < t_2 < \dots < t_k \leq n$ with*

$$(5.1) \quad \mu(E_{s_{t_1}} \cap E_{s_{t_2}} \cap \dots \cap E_{s_{t_k}}) \geq a^k - \epsilon.$$

Remark 5.9. To prove Proposition 5.10, one needs to use Proposition 5.8 repeatedly. To avoid explaining the same idea frequently, we illustrate how we will use Proposition 5.8 in the proof of Proposition 5.10 first.

Let $\{k_i^j\}_{i=1}^{\infty}$ be subsequences of \mathbb{Z} , $j \in \mathbb{N}$. Assume (X, \mathcal{B}, μ, T) is a measure preserving system and $A \in \mathcal{B}$ with positive measure. Let $A_1 = A$, $a_1 = \mu(A_1)$. We will show that there are $A_j \in \mathcal{B}$ and t_1^j, t_2^j, N_j such that $a_j = \mu(A_j) \geq \frac{1}{2}a_{j-1}^2 > 0$, and for $n \geq N_j$ and any tuple $\{s_1 < s_2 < \dots < s_n\}$ there exist $1 \leq t_1^j < t_2^j \leq n$ with $\mu(T^{-k_{s_{t_1^j}}^j} A_j \cap T^{-k_{s_{t_2^j}}^j} A_j) \geq \frac{1}{2}a_j^2$.

Let $E_i^1 = T^{-k_i^1} A$, $i \in \mathbb{N}$. Let $A_1 = A$, $a_1 = \mu(A_1)$ and let $N_1 = N(a_1, 2, \frac{1}{2}a_1^2)$ be as in Proposition 5.8. Then for $n \geq N_1$ and any tuple $\{s_1 < s_2 < \dots < s_n\}$ there exist $1 \leq t_1^1 < t_2^1 \leq n$ with $\mu(E_{s_{t_1^1}}^1 \cap E_{s_{t_2^1}}^1) \geq \frac{1}{2}a_1^2$.

Once one fixes a tuple $\{s_1 < s_2 < \cdots < s_n\}$, then one has a fixed $k_{t_1}^1$ and $k_{t_2}^1$ with $\mu(E_{k_{t_1}^1}^1 \cap E_{k_{t_2}^1}^1) \geq \frac{1}{2}a_1^2$. Now let $A_2 = A_1 \cap T^{-k_{t_2}^1 + k_{t_1}^1} A_1$, $a_2 = \mu(A_2) = \mu(E_{k_{t_1}^1}^1 \cap E_{k_{t_2}^1}^1) \geq \frac{1}{2}a_1^2$. Let $E_i^2 = T^{-k_i^2} A_2, i \in \mathbb{N}$. Let $N_2 = N(a_2, 2, \frac{1}{2}a_2^2)$ be as in Proposition 5.8. Thus for $n \geq N_2$ and any tuple $\{s_1 < s_2 < \cdots < s_n\}$ there exist $1 \leq t_1^2 < t_2^2 \leq n$ with $\mu(E_{s_{t_1}^2}^2 \cap E_{s_{t_2}^2}^2) \geq \frac{1}{2}a_2^2$. Then one fixes a tuple $\{s_1 < s_2 < \cdots < s_n\}$ and goes on as above.

Inductively, assume that $\{E_i^j = T^{-k_i^j} A_j\}_{i=1}^\infty, A_j, a_j, t_1^j, t_2^j, N_j$ are defined such that for $n \geq N_j$ and any tuple $\{s_1 < s_2 < \cdots < s_n\}$ there exist $1 \leq t_1^j < t_2^j \leq n$ with $\mu(E_{s_{t_1}^j}^j \cap E_{s_{t_2}^j}^j) \geq \frac{1}{2}a_j^2$. Fix a tuple $\{s_1 < s_2 < \cdots < s_n\}$, then one has a fixed $k_{t_1}^j$ and $k_{t_2}^j$ with $\mu(E_{k_{t_1}^j}^j \cap E_{k_{t_2}^j}^j) \geq \frac{1}{2}a_j^2$.

Let $A_{j+1} = A_j \cap T^{-k_{t_2}^j + k_{t_1}^j} A_j$ and $a_{j+1} = \mu(A_{j+1}) = \mu(E_{k_{t_1}^j}^j \cap E_{k_{t_2}^j}^j) \geq \frac{1}{2}a_j^2$. Let $E_i^{j+1} = T^{-k_i^{j+1}} A_{j+1}, i \in \mathbb{N}$, and let $N_{j+1} = N(a_{j+1}, 2, \frac{1}{2}a_{j+1}^2)$ be as in Proposition 5.8. Then for $n \geq N_{j+1}$ and any tuple $\{s_1 < s_2 < \cdots < s_n\}$ there exist $1 \leq t_1^{j+1} < t_2^{j+1} \leq n$ with $\mu(E_{s_{t_1}^{j+1}}^{j+1} \cap E_{s_{t_2}^{j+1}}^{j+1}) \geq \frac{1}{2}a_{j+1}^2$.

Note that the choices of $\{N_i\}$ is independent of $\{k_i^j\}_{i=1}^\infty$. \square

Now we are ready to show

Proposition 5.10. *The following statements hold.*

- (1) For each $d \in \mathbb{N}$, $\mathcal{F}_{fip} \subset \mathcal{F}_{P_d}$, which implies that $\mathcal{F}_{P_\infty} = \mathcal{F}_{fip}$.
- (2) $\mathcal{F}_{SG_d} \subset \mathcal{F}_{P_d}$ for each $d \in \mathbb{N} \cup \{\infty\}$. Moreover one has $\mathcal{F}_{fSG_d} \subset \mathcal{F}_{P_d}$.

Proof. (1) Let $F \in \mathcal{F}_{fip}$. Fix $d \in \mathbb{N}$. Now we show $F \in \mathcal{F}_{P_d}$. For this purpose, assume that (X, \mathcal{B}, μ, T) is a measure preserving system and $A \in \mathcal{B}$ with positive measure. Since $F \in \mathcal{F}_{fip}$, there are $p_1, p_2, \cdots, p_{\ell_d} \in \mathbb{Z}$ with $\ell_d = d \sum_{i=1}^d N_i$ such that $F \supset FS\{p_i\}_{i=1}^{\ell_d}$, where N_i are chosen as in Remark 5.9 for (X, \mathcal{B}, μ, T) and A .

Let $A_1 = A$. For $p_1, p_1 + p_2, \cdots, p_1 + \cdots + p_{N_1}$ by the argument in Remark 5.9 there is $q_1 = p_{i_1}^1 + \cdots + p_{i_2}^1$ such that $\mu(A_1 \cap T^{-q_1} A_1) \geq \frac{1}{2}a_1^2$, where $a_1 = \mu(A_1)$ and $1 \leq i_1^1 < i_2^1 \leq N_1$. Let $A_2 = A_1 \cap T^{-q_1} A_1$ and $a_2 = \mu(A_2)$. For $p_{N_1+1}, p_{N_1+1} + p_{N_1+2}, \cdots, p_{N_1+1} + \cdots + p_{N_1+N_2}$, there is $q_2 = p_{i_1}^2 + \cdots + p_{i_2}^2$ such that $\mu(A_2 \cap T^{-q_2} A_2) \geq \frac{1}{2}a_2^2$, where $N_1 + 1 \leq i_1^2 < i_2^2 \leq N_1 + N_2$. Note that $q_1, q_2, q_1 + q_2 \in F$.

Inductively we obtain

$$N_1 + \cdots + N_j + 1 \leq i_1^{j+1} < i_2^{j+1} \leq N_1 + \cdots + N_{j+1}, \quad 0 \leq j \leq d-1.$$

q_1, \dots, q_d and A_1, \dots, A_q with $q_j = \sum_{i=i_1^j}^{i_2^j} p_i$ and $A_j = A_{j-1} \cap T^{-q_{j-1}} A_{j-1}$, $a_j = \mu(A_j)$ such that $\mu(A_j \cap T^{-q_j} A_j) \geq \frac{1}{2} a_j^2$. Thus

$$\mu(A \cap \bigcap_{n \in FS(\{q_i\}_{i=1}^d)} T^{-n} A) \geq \frac{1}{2} a_d^2 > 0,$$

and it is clear that $F \supset FS(\{q_i\}_{i=1}^d)$. This implies that $F \in \mathcal{F}_{P_d}$.

Thus $\mathcal{F}_{P_\infty} \supset \mathcal{F}_{fip}$. Since it is clear that $\mathcal{F}_{P_\infty} \subset \mathcal{F}_{fip}$, we are done.

(2) Since each SG_1 -set is a Δ -set, and hence it is a Poincaré sequence (this is easy to be checked by Poincaré recurrence Theorem [15]). We First show the case when $d = 2$ which will illustrate the general idea. Then we give the proof for the general case.

Let $F \in SG_2$. Then there is $P = \{P_i\}_{i=1}^\infty \subset \mathbb{Z}$ with $F = SG_2(P)$. Let (X, \mathcal{B}, μ, T) be a m.d.s. and $A \in \mathcal{B}$ with $\mu(A) > 0$. Set $A_1 = A$ and $a_1 = \mu(A_1)$.

Let

$$q_1 = \sum_{i=1}^{N_2} P_{2i-1}, q_2 = \sum_{i=N_2+1}^{2N_2} P_{2i-1}, \dots, \text{ and } q_{N_1} = \sum_{i=(N_1-1)N_2+1}^{N_1 N_2} P_{2i-1},$$

where $N_1 = N(a_1, 2, \frac{1}{2} a_1^2)$ and $N_2 = N(a_2, 2, \frac{1}{2} a_2^2)$ are chosen as in Remark 5.9 for (X, \mathcal{B}, μ, T) and A . Consider the sequence $q_1, q_1 + q_2, \dots, q_1 + q_2 + \dots + q_{N_1}$. Then as in Remark 5.9 there are $1 \leq i_1, j_1 \leq N_1$ such that $\mu(A_2) \geq \frac{1}{2} \mu(A)^2$, where $A_2 = A_1 \cap T^{-n_1} A_1$ and $n_1 = \sum_{i=i_1}^{j_1} q_i$. Note that

$$n_1 = P_{2(i_1-1)N_2+1} + P_{2(i_1-1)N_2+3} + \dots + P_{2j_1 N_2-1}.$$

Now consider the sequence

$$P_{2(i_1-1)N_2}, P_{2(i_1-1)N_2} + P_{2(i_1-1)N_2+2}, \dots, P_{2(i_1-1)N_2} + P_{2(i_1-1)N_2+2} + \dots + P_{2j_1 N_2}.$$

It has N_2 terms. So as in Remark 5.9 there are $1 \leq i_2, j_2 \leq N_2$ such that $\mu(A_2 \cap T^{-n_2} A_2) > 0$, where $n_2 = \sum_{i=(i_1-1)N_2+i_2}^{(i_1-1)N_2+j_2} P_{2i}$. Note that $n_1, n_2, n_1 + n_2 \in F$ by the definition of $SG_2(P)$. It is easy to verify that

$$\mu(A \cap T^{-n_1} A \cap T^{-n_2} A \cap T^{-n_1-n_2} A) \geq \frac{1}{2} \mu(A_2)^2 > 0.$$

Hence $F \in \mathcal{F}_{P_2}$.

Now we show the general case. Assume that $d \geq 3$ and let $F \in SG_d$. We show that $F \in \mathcal{F}_{P_d}$.

Since $F \in SG_d$, there is $P = \{P_i\}_{i=1}^\infty \subset \mathbb{Z}$ with $F = SG_d(P)$. Let (X, \mathcal{B}, μ, T) be a measure preserving system and $A \in \mathcal{B}$ with $\mu(A) > 0$. Set $A_1 = A$. Let N_1, \dots, N_d be the numbers as defined in Remark 5.9 for (X, \mathcal{B}, μ, T) , A and let $M_i = \prod_{j=i}^d N_j$ for $1 \leq i \leq d$.

Let

$$q_1^1 = \sum_{i=1}^{M_2} P_{di-(d-1)}, q_2^1 = \sum_{i=M_2+1}^{2M_2} P_{di-(d-1)}, \dots, q_{N_1}^1 = \sum_{i=(N_1-1)M_2+1}^{M_1} P_{di-(d-1)}.$$

Consider the sequence $q_1^1, q_1^1 + q_2^1, \dots, q_1^1 + q_2^1 + \dots + q_{N_1}^1$. Then as in Remark 5.9 there are $1 \leq i_1, j_1 \leq N_1$ such that $\mu(A_2) \geq \frac{1}{2}\mu(A_1)^2$, where $A_2 = A_1 \cap T^{-n_1}A_1$ and $n_1 = \sum_{i=i_1}^{j_1} q_i^1$.

Let $m_1 = (i_1 - 1)M_2$. Note that there is $t_1 \geq M_2 - 1$ such that

$$n_1 = \sum_{i=i_1}^{j_1} q_i^1 = P_{dm_1+1} + P_{dm_1+d+1} + \dots + P_{dm_1+t_1d+1}.$$

Now consider

$$q_1^2 = \sum_{i=m_1+1}^{m_1+M_3} P_{di-(d-2)}, q_2^2 = \sum_{i=m_1+M_3+1}^{m_1+2M_3} P_{di-(d-2)}, \dots, q_{N_2}^2 = \sum_{i=m_1+(N_2-1)M_3+1}^{m_1+M_2} P_{di-(d-2)}.$$

Now consider $q_1^2, q_1^2 + q_2^2, \dots, q_1^2 + q_2^2 + \dots + q_{N_2}^2$. It has N_2 terms. So as in Remark 5.9 there are $1 \leq i_2, j_2 \leq N_2$ such that $\mu(A_3) \geq \frac{1}{2}\mu(A_2)^2$, where $A_3 = A_2 \cap T^{-n_2}A_2$ and $n_2 = \sum_{i=i_2}^{j_2} q_i^2$. Let $m_2 = m_1 + (i_2 - 1)M_3$. Note that $n_1, n_2, n_1 + n_2 \in F$ and there is $t_2 \geq M_3 - 1$ such that

$$n_2 = \sum_{i=i_2}^{j_2} q_i^2 = P_{dm_2+2} + P_{dm_2+d+2} + \dots + P_{dm_2+t_2d+2}.$$

Note that n_2 has at least M_3 terms.

Inductively for $1 \leq k \leq d - 1$ we have $1 \leq i_k, j_k \leq N_k$ and

$$n_k = \sum_{i=i_k}^{j_k} q_i^k = P_{dm_k+k} + P_{dm_k+d+k} + \dots + P_{dm_k+t_kd+k},$$

where $t_k \geq M_{k+1} - 1$. Also we have $A_k = A_{k-1} \cap T^{-n_{k-1}}A_{k-1}$ with $\mu(A_k) \geq \frac{1}{2}\mu(A_{k-1})^2$, and $FS(\{n_j\}_{j=1}^k) \subset F$.

Especially, when $k = d$, we get $1 \leq i_d < j_d \leq N_d$ and $n_d = \sum_{i=i_d}^{j_d} P_{di}$. By the definition of SG_d we get that $FS(\{n_i\}_{i=1}^d) \subset F$. From the definition of $A_j, j = 1, 2, \dots, d$, one has

$$\mu(A \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}A) \geq \frac{1}{2}\mu(A_d)^2 > 0,$$

which implies that $F \in \mathcal{F}_{P_d}$. The proof is completed. \square

5.3.3. *Proof of Proposition 5.5:* It is clear that $\mathcal{F}_{B_\infty} \subset \mathcal{F}_{fip}$. Since $\mathcal{F}_{fip} \subset \mathcal{F}_{P_\infty} \subset \mathcal{F}_{B_\infty}$ (by Proposition 5.10 and the obvious fact that $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d}$) we have $\mathcal{F}_{B_\infty} = \mathcal{F}_{fip}$.

5.4. Poincaré recurrence sets and $\mathbf{RP}^{[d]}$.

Theorem 5.11. *Let (X, T) be a minimal t.d.s.. Then for each $d \in \mathbb{N} \cup \{\infty\}$, $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{P_d}$ for any neighborhood U of y .*

Proof. We first show the case when $d \in \mathbb{N}$. (\Leftarrow) Since $\mathcal{F}_{P_d} \subset \mathcal{F}_{B_d}$, it follows from Theorem 5.5. Or one proves it directly as follows. Assume $N(x, U) \in \mathcal{F}_{P_d}$ for any neighborhood U of y and $\mu \in M(X, T)$. Then $\text{supp}(\mu) = X$ since (X, T) is minimal. For any $\epsilon > 0$, let $U_1 = B(x, \epsilon)$ and $U_2 = B(y, \frac{\epsilon}{2})$. Since $N(x, U_2)$ is a Poincaré sequence of order d and $\mu(U_2) > 0$ there exist $n_1, \dots, n_d \in \mathbb{Z}$ such that $T^{n_i}x \in U_2$ for $n \in FS(\{n_i\}_{i=1}^d)$ and $\mu(U_2 \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U_2) > 0$. Then any $y' \in U_2 \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U_2$ satisfies $T^n y' \in U_2$ for any $n \in FS(\{n_i\}_{i=1}^d)$. Thus, $\rho(y, y') < \epsilon$ and $\rho(T^m x, T^m y') \leq \text{diam}(U_2) < \epsilon$ for any $n \in FS(\{n_i\}_{i=1}^d)$, which imply that $(x, y) \in \mathbf{RP}^{[d]}$.

(\Rightarrow) Assume that $(x, y) \in \mathbf{RP}^{[d]}$ and U is a neighborhood of y . By Theorem 4.3, $N(x, U) \in \mathcal{F}_{SG_d}$. Then by Proposition 5.10 we have $N(x, U) \in \mathcal{F}_{P_d}$.

The case $d = \infty$ follows from the case $d \in \mathbb{N}$ and definitions. \square

5.5. Conclusion. Now we sum up the results of this section and previous two sections. Note that \mathcal{F}_{Bir_∞} and \mathcal{F}_{Poi_∞} can be defined naturally. Since $\mathcal{F}_{1,0} \subset \mathcal{F}_{2,0} \subset \dots$ we define $\mathcal{F}_{\infty,0} =: \bigcup_{d=1}^{\infty} \mathcal{F}_{d,0}$. Another way to do this is that one follows the idea in [10] to define ∞ -step nilsystems and view $\mathcal{F}_{\infty,0}$ as the family generated by all Nil_∞ Bohr₀-sets. It is easy to check that Theorem 3.12 holds for $d = \infty$.

Thus we have

Theorem 5.12. *Let (X, T) be a minimal t.d.s. and $x, y \in X$. Then the following statements are equivalent for $d \in \mathbb{N} \cup \{\infty\}$:*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$.
- (2) $N(x, U) \in \mathcal{F}_{d,0}^*$ for each neighborhood U of y .
- (3) $N(x, U) \in \mathcal{F}_{Poi_d}$ for each neighborhood U of y .
- (4) $N(x, U) \in \mathcal{F}_{Bir_d}$ for each neighborhood U of y .
- (5) $N(x, U) \in \mathcal{F}_{SG_d}$ for each neighborhood U of y .
- (6) $N(x, U) \in \mathcal{F}_{fSG_d}$ for each neighborhood U of y .
- (7) $N(x, U) \in \mathcal{F}_{B_d}$ for each neighborhood U of y .
- (8) $N(x, U) \in \mathcal{F}_{P_d}$ for each neighborhood U of y .

6. d -STEP ALMOST AUTOMORPHY AND RECURRENCE SETS

In the previous sections we give some characterizations of regionally proximal relation of order d . In the present section we introduce and study d -step almost automorphy.

6.1. Definition of d -step almost automorphy.

6.1.1. First we recall the notion of d -step almost automorphic systems and give its structure theorem.

Definition 6.1. Let (X, T) be a t.d.s. and $x \in X$, $d \in \mathbb{N} \cup \{\infty\}$. x is called an d -step almost automorphic point (or d -step AA for short) if $\mathbf{RP}^{[d]}(Y)[x] = \{x\}$, where $Y = \overline{\{T^n x : n \in \mathbb{Z}\}}$ and $\mathbf{RP}^{[d]}(Y)[x] = \{y \in Y : (x, y) \in \mathbf{RP}^{[d]}(Y)\}$.

A minimal t.d.s. (X, T) is called d -step almost automorphic if it has a d -step almost automorphic point.

Remark 6.2. Since

$$\mathbf{RP}^{[\infty]} \subseteq \dots \subseteq \mathbf{RP}^{[d]} \subseteq \mathbf{RP}^{[d-1]} \subseteq \dots \subseteq \mathbf{RP}^{[1]},$$

we have

$$\text{AA} = 1\text{-step AA} \Rightarrow \dots \Rightarrow (d-1)\text{-step AA} \Rightarrow d\text{-step AA} \Rightarrow \dots \Rightarrow \infty\text{-step AA}.$$

6.1.2. The following theorem follows from Theorem 2.5.

Theorem 6.3. *Let (X, T) be a minimal t.d.s.. Then (X, T) is a d -step almost automorphic system for some $d \in \mathbb{N} \cup \{\infty\}$ if and only if it is an almost one-to-one extension of its maximal d -step nilfactor (X_d, T) .*

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \pi \downarrow & & \downarrow \pi \\ X_d & \xrightarrow{T} & X_d \end{array}$$

6.2. **1-step almost automorphy.** First we recall some classical results about almost automorphy.

Let (X, T) be a minimal t.d.s.. In [39] it is proved that $(x, y) \in \mathbf{RP}^{[1]}$ if and only if for each neighborhood U of y , $N(x, U)$ contains some Δ -set, see also Theorem 4.3. Similarly, we have for a minimal system (X, T) , $(x, y) \in \mathbf{RP}^{[1]}$ if and only if for each neighborhood U of y , $N(x, U) \in \mathcal{F}_{Poi}$ [31], see also Theorem 5.12.

Using these theorems and the facts that \mathcal{F}_{Poi} and \mathcal{F}_{Bir} have the Ramsey property, one has

Theorem 6.4. *Let (X, T) be a minimal t.d.s. and $x \in X$. Then the following statements are equivalent:*

- (1) x is AA.
- (2) $N(x, V) \in \mathcal{F}_{Poi}^*$ for each neighborhood V of x .
- (3) $N(x, V) \in \mathcal{F}_{Bir}^*$ for each neighborhood V of x .
- (4) [16] $N(x, V) \in \Delta^*$ for each neighborhood V of x .

We will not give the proof of this theorem since it is the special case of Theorem 6.8.

6.3. ∞ -step almost automorphy. In this subsection we give one characterization for ∞ -step AA. Followed from Theorem 2.5, one has

Theorem 6.5. *Let (X, T) be a minimal t.d.s. and $d \geq 1$. Then*

- (1) $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U)$ contains a finite IP-set of length $d + 1$ for any neighborhood U of y , and thus
- (2) $(x, y) \in \mathbf{RP}^{[\infty]}$ if and only if $N(x, U) \in \mathcal{F}_{fip}$ for any neighborhood U of y .

To show the next theorem we need the following lemma which should be known, see for example Huang, Li and Ye [30].

Lemma 6.6. \mathcal{F}_{fip} has the Ramsey property.

We have the following

Theorem 6.7. *Let (X, T) be a minimal t.d.s.. Then (X, T) is ∞ -step AA if and only if there is $x \in X$ such that $N(x, V) \in \mathcal{F}_{fip}^*$ for each neighborhood V of x .*

Proof. Assume that there is $x \in X$ such that $N(x, V) \in \mathcal{F}_{fip}^*$ for each neighborhood V of x . If there is $y \in X$ such that $(x, y) \in \mathbf{RP}^{[\infty]}$, then by Proposition 6.5 for any neighborhood U of y , $N(x, U) \in \mathcal{F}_{fip}$. This implies that $x = y$, i.e. (X, T) is ∞ -step AA.

Now assume that (X, T) is ∞ -step AA, i.e. there is $x \in X$ such that $\mathbf{RP}^{[\infty]}[x] = \{x\}$. If for some neighborhood V of x , $N(x, V) \notin \mathcal{F}_{fip}^*$, then $N(x, V^c)$ contains finite IP-sets of arbitrarily long lengths.

Let $U_1 = V^c$. Covering U_1 by finitely many closed balls $U_1^1, \dots, U_1^{i_1}$ of diam ≤ 1 . Then there is j_1 such that $N(x, U_1^{j_1})$ contains finite IP-sets of arbitrarily long lengths. Let $U_2 = U_1^{j_1}$. Covering U_2 by finitely many closed balls $U_2^1, \dots, U_2^{i_2}$ of diam $\leq \frac{1}{2}$. Then there is j_2 such that $N(x, U_2^{j_2})$ contains finite IP-sets of arbitrarily long lengths. Let $U_3 = U_2^{j_2}$. Inductively, there are a sequence of closed balls U_n with diam $\leq \frac{1}{n}$ such that $N(x, U_n)$ contains finite IP-sets of arbitrarily long lengths. Let $\{y\} = \bigcap U_n$. It is clear that $(x, y) \in \mathbf{RP}^{[\infty]}$ with $y \neq x$, a contradiction. Thus $N(x, V) \in \mathcal{F}_{fip}^*$ for each neighborhood V of x . \square

6.4. Characterization of d -step almost automorphy. Now we use the results built in previous sections to get the following characterization for d -step AA via recurrence sets.

Theorem 6.8. *Let (X, T) be a minimal t.d.s., $x \in X$ and $d \in \mathbb{N} \cup \{\infty\}$. Then the following statements are equivalent:*

- (1) x is d -step AA point.
- (2) $N(x, V) \in \mathcal{F}_{d,0}$ for each neighborhood V of x .
- (3) $N(x, V) \in \mathcal{F}_{Poi_d}^*$ for each neighborhood V of x .
- (4) $N(x, V) \in \mathcal{F}_{Bir_d}^*$ for each neighborhood V of x .

Proof. Roughly speaking the theorem follows from Theorem 5.12, the fact $\mathcal{F}_{d,0}^*$, \mathcal{F}_{Poi_d} and \mathcal{F}_{Bir_d} have the Ramsey property, and the idea of the proof of Theorem 6.7. We show that (1) \Leftrightarrow (2), and the rest is similar.

(1) \Rightarrow (2): Let x be a d -step AA point. If (2) does not hold, then there is some neighborhood V of x such that $N(x, V) \notin \mathcal{F}_{d,0}$. Then $N(x, V^c) = \mathbb{Z} \setminus N(x, V) \in \mathcal{F}_{d,0}^*$. Since $\mathcal{F}_{d,0}^*$ has the Ramsey property, similar to the proof of Theorem 6.7 one can find some $y \in V^c$ such that $N(x, U) \in \mathcal{F}_{d,0}^*$ for every neighborhood U of y . By Theorem 5.12, $y \in \mathbf{RP}^{[d]}[x]$. Since $y \neq x$, this contradicts to the fact x being d -step AA.

(2) \Rightarrow (1): If x is not d -step AA, then there is some $y \in \mathbf{RP}^{[d]}[x]$ with $x \neq y$. Let U_x and U_y be neighborhoods of x and y with $U_x \cap U_y = \emptyset$. By (2) $N(x, U_x) \in \mathcal{F}_{d,0}$. By Theorem 5.12, $N(x, U_y) \in \mathcal{F}_{d,0}^*$. Hence $N(x, U_x) \cap N(x, U_y) \neq \emptyset$, which contradicts the fact that $U_x \cap U_y = \emptyset$. \square

6.5. Some further questions. (1) We have defined and studied d -recurrence and Poincaré sequence of order d ; and d -topological recurrence and Birkhoff recurrence set of order d . It is not clear the relation between \mathcal{F}_{P_d} and \mathcal{F}_{Poi_d} ; and \mathcal{F}_{B_d} and \mathcal{F}_{Bir_d} . Also it will be very interesting if one can show that $\mathcal{F}_{B_d} \subset \mathcal{F}_{d,0}^*$ which implies that x is d -step AA if and only if $N(x, V) \in \mathcal{F}_{B_d}^*$ for each neighborhood V of x by Theorem 6.8.

(2) In [38] Veech proved that for a minimal t.d.s. (X, T) , a point $x \in X$ is almost automorphic if and only if from any sequence $\{n'_i\} \subseteq \mathbb{Z}$ one may extract a subsequence $\{n_i\}$ such that $\lim_{i \rightarrow \infty} T^{n_i}x = y$ for some $y \in X$ and $\lim_{i \rightarrow \infty} T^{-n_i}y = x$. We do not know if there is a similar characterization for d -step almost automorphic points for $d \geq 2$.

APPENDIX A. THE RAMSEY PROPERTIES

Recall that a family \mathcal{F} has the Ramsey property means that if $A \in \mathcal{F}$ and $A = \cup_{i=1}^n A_i$ then one of A_i is still in \mathcal{F} . In this section, we show that \mathcal{F}_{SG_2} does not have the Ramsey property.

Theorem A.1. \mathcal{F}_{SG_2} does not have the Ramsey property.

Proof. Let $P = \{p_1, p_2, \dots\}$ be a subsequence of \mathbb{N} with $p_{i+1} > 2(p_1 + \dots + p_i)$. The assumption that $p_{i+1} > 2(p_1 + \dots + p_i)$ ensures that each element of $SG_2(P)$ has a unique expression with the form of $\sum_i p_j$.

Now divide the set $SG_2(P)$ into the following three sets:

$$\begin{aligned} B_1 &= \{p_{2n-1} + \dots + p_{2m-1} : n \leq m \in \mathbb{N}\} = SG_1(\{p_1, p_3, \dots\}), \\ B_2 &= \{p_{2n} + \dots + p_{2m} : n \leq m \in \mathbb{N}\} = SG_1(\{p_2, p_4, \dots\}), \\ B_0 &= SG_2(P) \setminus (B_1 \cup B_2). \end{aligned}$$

We show that $B_i \notin \mathcal{F}_{SG_2}$ for $i = 0, 1, 2$. In fact, we will prove that for each $i = 0, 1, 2$ there do not exist $a_1 < a_2 < a_3$ such that

$$(*) \quad a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \subseteq B_i,$$

which obviously implies that $B_i \notin \mathcal{F}_{SG_2}$ for $i = 0, 1, 2$.

(1). First we show $B_2 \notin \mathcal{F}_{SG_2}$. The proof $B_1 \notin \mathcal{F}_{SG_2}$ follows similarly. Assume the contrary, i.e. there exist $a_1 < a_2 < a_3$ such that

$$a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \subseteq B_2.$$

Let

$$\begin{aligned} a_1 &= p_{2n_1} + \dots + p_{2m_1}, \quad n_1 \leq m_1; \\ a_2 &= p_{2n_2} + \dots + p_{2m_2}, \quad n_2 \leq m_2; \\ a_3 &= p_{2n_3} + \dots + p_{2m_3}, \quad n_3 \leq m_3. \end{aligned}$$

Since $a_1 < a_2 < a_3$ and the assumption that $p_{i+1} > 2(p_1 + \dots + p_i)$, one has that $m_1 \leq m_2 \leq m_3$. Since $a_1 + a_2, a_2 + a_3 \in B_2$, one has that $n_2 = m_1 + 1$ and $n_3 = m_2 + 1$. Hence $n_3 = m_2 + 1 \geq n_2 + 1 = m_1 + 2$, i.e. $n_3 > m_1 + 1$. Thus

$$a_1 + a_3 \notin B_2,$$

a contraction!

(2). Now we show $B_0 \notin \mathcal{F}_{SG_2}$. Assume the contrary, i.e. there exist $a_1 < a_2 < a_3$ such that

$$a_1, a_2, a_3, a_1 + a_2, a_2 + a_3, a_1 + a_3 \subseteq B_0.$$

Let

$$\begin{aligned} a_1 &= p_{i_1^1} + p_{i_2^1} + \dots + p_{i_{k_1}^1}; \\ a_2 &= p_{i_1^2} + p_{i_2^2} + \dots + p_{i_{k_2}^2}; \\ a_3 &= p_{i_1^3} + p_{i_2^3} + \dots + p_{i_{k_3}^3}, \end{aligned}$$

where $i_1^r < i_2^r < \dots < i_{k_r}^r$, $i_{j+1}^r \leq i_j^r + 2$ for $1 \leq j \leq k_r - 1$, and there are both even and odd numbers in $\{i_1^r, i_2^r, \dots, i_{k_r}^r\}$ ($r = 1, 2, 3$).

Since there are both even and odd numbers in $\{i_1^r, i_2^r, \dots, i_{k_r}^r\}$ ($r = 1, 2, 3$) and $i_{j+1}^r \leq i_j^r + 2$ for $1 \leq j \leq k_r - 1$, there exist $1 \leq j_r \leq k_r - 1$ such that $i_{j_r+1}^r = i_{j_r}^r + 1$. Since $a_1 < a_2 < a_3$ and the assumption that $p_{i+1} > 2(p_1 + \dots + p_i)$, one has that $i_{k_1}^1 \leq i_{k_2}^2 \leq i_{k_3}^3$. Note that we have

$$\begin{aligned} i_1^1 &< i_2^1 < \dots < i_{j_1}^1 < i_{j_1+1}^1 = i_{j_1}^1 + 1 < \dots < i_{k_1}^1, \\ i_1^2 &< i_2^2 < \dots < i_{j_2}^2 < i_{j_2+1}^2 = i_{j_2}^2 + 1 < \dots < i_{k_2}^2, \\ i_1^3 &< i_2^3 < \dots < i_{j_3}^3 < i_{j_3+1}^3 = i_{j_3}^3 + 1 < \dots < i_{k_3}^3. \end{aligned}$$

The condition $a_1 + a_2 \in B_0$ implies that

$$(a) \quad i_{j_1+1}^1 < i_1^2 \leq i_{k_1}^1 + 2; \quad i_{k_1}^1 < i_{j_2}^2.$$

In fact if $i_1^2 < i_{j_1}^1$, then the gap $\{i_{j_1}^1, i_{j_1}^1 + 1\}$ is missing in the term of a_2 and it contradicts the assumption $a_2 \in \mathcal{F}_{SG_2}$. The statement $i_{k_1}^1 < i_{j_2}^2$ follows by the same argument.

Similarly, using the assumptions $a_2 + a_3 \in B_0$ and $a_1 + a_3 \in B_0$, one has

$$(b) \quad i_{j_2+1}^2 < i_1^3 \leq i_{k_2}^2 + 2; \quad i_{k_2}^2 < i_{j_3}^3.$$

and

$$(c) \quad i_{j_1+1}^1 < i_1^3 \leq i_{k_1}^1 + 2; \quad i_{k_1}^1 < i_{j_3}^3.$$

From (a), we have that $i_{k_1}^1 < i_{j_2}^2$; and from (b), we have $i_{j_2+1}^2 = i_{j_2}^2 + 1 < i_1^3$. Hence we have $i_1^3 \geq i_{k_1}^1 + 3$, which contradicts (c). The proof is completed. \square

APPENDIX B. COMPACT HAUSDORFF SYSTEMS

In this section we discuss compact Hausdorff systems, i.e. the systems with phase space being compact Hausdorff. The reason for this is not generalization for generalization's sake, but rather that we have to deal with non-metrizable systems. For example, we will use (in the proof of Theorem 3.5) an important tool named Ellis semigroup which is a subspace of an uncountable product of copies of the phase space and therefore in general not metrizable.

B.1. Compact Hausdorff systems. In the classical theory of abstract topological dynamics, the basic assumption about the system is that the space is a compact Hausdorff space and the action group is a topological group. In this paper, we mainly consider the compact metrizable system under \mathbb{Z} -actions, but in some occasions we have to deal with compact Hausdorff spaces which are non-metrizable. Note that each compact Hausdorff space is a uniform space, and one may use the uniform structure replacing the role of a metric, see for example the Appendix of [2].

First we recall a classical equality concerning regionally proximal relation in compact Hausdorff systems. A *compact Hausdorff system* is a pair (X, T) , where X is a compact Hausdorff space and $T : X \rightarrow X$ is a homeomorphism. Let (X, T) be a compact Hausdorff system and \mathcal{U}_X be the unique uniform structure of X . The *regionally proximal relation* on X is defined by

$$\mathbf{RP} = \bigcap_{\alpha \in \mathcal{U}_X} \overline{\bigcup_{n \in \mathbb{Z}} (T \times T)^{-n} \alpha}$$

B.2. Ellis semigroup. A beautiful characterization of distality was given by R. Ellis using so-called enveloping semigroup. Given a compact Hausdorff system (X, T) , its *enveloping semigroup* (or *Ellis semigroup*) $E(X, T)$ is defined as the closure of the set $\{T^n : n \in \mathbb{Z}\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). Ellis showed that a compact Hausdorff system (X, T) is distal if and only if $E(X, T)$ is a group if and only if every point in $(X^2, T \times T)$ is minimal [11].

B.3. Limits of Inverse systems. Suppose that every λ in a set Λ directed by the relation \leq corresponds a t.d.s. (X_λ, T_λ) , and that for any $\lambda, \xi \in \Lambda$ satisfying $\xi \leq \lambda$ a factor map $\pi_\xi^\lambda : (X_\lambda, T_\lambda) \rightarrow (X_\xi, T_\xi)$ is defined; suppose further that $\pi_\tau^\xi \pi_\xi^\lambda = \pi_\tau^\lambda$ for all $\lambda, \xi, \tau \in \Lambda$ with $\tau \leq \xi \leq \lambda$ and that $\pi_\lambda^\lambda = \text{id}_X$ for all $\lambda \in \Lambda$. In this situation we say that the family $\{(X_\lambda, T_\lambda), \pi_\xi^\lambda, \Lambda\} = \{(X_\lambda, T_\lambda), \pi_\xi^\lambda, \Lambda\}$ is an *inverse system of the systems* (X_λ, T_λ) ; and the mappings π_ξ^λ are called *bonding mappings* of the inverse system.

Let $\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$ be an inverse system. The *limit of the inverse system* $\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$ is the set

$$\left\{ (x_\lambda)_\lambda \in \prod_{\lambda \in \Lambda} X_\lambda : \pi_\xi^\lambda(x_\lambda) = x_\xi \text{ for all } \xi \leq \lambda \in \Lambda \right\},$$

and is denoted by $\varprojlim\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$. Let $X = \varprojlim\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$. For each $\lambda \in \Lambda$, let $\pi_\lambda : X \rightarrow X_\lambda, (x_\sigma)_\sigma \mapsto x_\lambda$ be the projection mapping.

A well known result is the following (see for example [34]):

Lemma B.1. *Each compact Hausdorff system is the inverse limit of topological dynamical systems.*

B.4. The regionally proximal relation of order d for compact Hausdorff systems. The definition of the regionally proximal relation of order d for compact Hausdorff systems is similar to the metric case.

Definition B.2. Let (X, T) be a compact Hausdorff system, \mathcal{U}_X be the unique uniform structure of X and let $d \geq 1$ be an integer. A pair $(x, y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\alpha \in \mathcal{U}_X$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ such that $(x, x') \in \alpha, (y, y') \in \alpha$, and

$$(T^{\mathbf{n} \cdot \epsilon} x', T^{\mathbf{n} \cdot \epsilon} y') \in \alpha \text{ for any } \epsilon \in \{0, 1\}^d, \epsilon \neq (0, \dots, 0),$$

where $\mathbf{n} \cdot \epsilon = \sum_{i=1}^d \epsilon_i n_i$. The set of all regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order d* .

By Lemma B.1, each compact Hausdorff system is the inverse limit of topological dynamical systems. Recall the definition of the product uniformity. Let $(X_\lambda, \mathcal{U}_\lambda)_{\lambda \in \Lambda}$ be a family of uniform spaces and let $Z = \prod_{\lambda \in \Lambda} X_\lambda$. The uniformity on Z (the product uniformity) is defined as follows. If $F = \{\lambda_1, \dots, \lambda_m\}$ is a finite subset of the index set Λ and $\alpha_{\lambda_j} \in \mathcal{U}_{\lambda_j}$ ($j = 1, \dots, m$), let

$$\Phi_{\alpha_{\lambda_1}, \dots, \alpha_{\lambda_m}} = \{(x, y) \in Z \times Z : (x_{\lambda_j}, y_{\lambda_j}) \in \alpha_{\lambda_j}, j = 1, \dots, m\}.$$

The collection of all such sets $\Phi_{\alpha_{\lambda_1}, \dots, \alpha_{\lambda_m}}$ for all finite subsets F of Λ is a base for the product uniformity. From this and the definition of the regionally proximal relation of order d , one has the following result.

Proposition B.3. *Let (X, T) be a compact Hausdorff system and $d \in \mathbb{N}$. Suppose that $X = \varprojlim\{X_\lambda, \pi_\xi^\lambda, \Lambda\}$, where $(X_\lambda, T_\lambda)_{\lambda \in \Lambda}$ are t.d.s.. Then*

$$\mathbf{RP}^{[d]}(X) = \varprojlim\{\mathbf{RP}^{[d]}(X_\lambda), \pi_\xi^\lambda \times \pi_\xi^\lambda, \Lambda\}.$$

Thus combining this proposition with Theorem 2.5, one has

Theorem B.4. *Let (X, T) be a minimal compact Hausdorff system and $d \in \mathbb{N}$. Then*

- (1) $\mathbf{RP}^{[d]}(X)$ is an equivalence relation, and so is $\mathbf{RP}^{[\infty]}$.
- (2) If $\pi : (X, T) \rightarrow (Y, S)$ is a factor map, then $(\pi \times \pi)(\mathbf{RP}^{[d]}(X)) = \mathbf{RP}^{[d]}(Y)$.
- (3) $(X/\mathbf{RP}^{[d]}, T)$ is the maximal nilfactor of (X, T) .

Note that for a compact Hausdorff system (X, T) we say that it is a system of order d for some $d \in \mathbb{N}$ if it is an inverse limit of basic d -step nilsystems.

APPENDIX C. INTERSECTIVE

It is well known that P is a Birkhoff recurrence set iff $P \cap (F - F) \neq \emptyset$ for each $F \in \mathcal{F}_s$. To give a similar characterization we have

Definition C.1. A subset P is intersective (topologically) of order d if for each $F \in \mathcal{F}_s$ there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ and $a \in F$ with $a + FS(\{n_i\}_{i=1}^d) \subset F$, i.e. $F \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} (F - n) \neq \emptyset$.

Theorem C.2. A subset P is intersective (topologically) of order d if and only if it is a Birkhoff recurrence set of order d .

Proof. Assume that P is a Birkhoff recurrence set of order d . Let $F \in \mathcal{F}_s$. Then $1_F \in \{0, 1\}^{\mathbb{Z}^+}$. Let (X, T) be a minimal subsystem of $(\overline{orb(1_F, T)}, T)$, where T is the shift. Since $F \in \mathcal{F}_s$, $[1]$ is a non-empty open subset of X . By the definition there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ such that $[1] \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}[1] \right) \neq \emptyset$. It implies that there is $a \in F$ with $a + FS(\{n_i\}_{i=1}^d) \subset F$ and hence P is intersective (topologically) of order d .

Assume that P is intersective (topologically) of order d . Let (X, T) be a minimal t.d.s. and U be an open non-empty subsets. Take $x \in U$, then $F = N(x, U) \in \mathcal{F}_s$. Thus there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ and $a \in F$ with $a + FS(\{n_i\}_{i=1}^d) \subset F$. It follows that $U \cap \left(\bigcap_{n \in FS(\{n_i\}_{i=1}^d)} T^{-n}U \right) \neq \emptyset$. \square

It is well known that P is a Poincaré sequence if and only if $P \cap (F - F) \neq \emptyset$ for each $F \in \mathcal{F}_{pubd}$. To give a similar characterization we have

Definition C.3. A subset P is intersective of order d if for each $F \in \mathcal{F}_{pubd}$ there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ and $a \in F$ with $a + FS(\{n_i\}_{i=1}^d) \subset F$.

Theorem C.4. A subset is intersective of order d if and only if it is a Poincaré sequence of order d .

Proof. Assume that P is intersective of order d . Let (X, \mathcal{B}, μ, T) be a measure preserving system and $A \in \mathcal{B}$ with $\mu(A) > 0$. By the Furstenberg corresponding principle, there exists $F \subset \mathbb{Z}$ such that $d(F) \geq \mu(A)$ and

$$(C.1) \quad \left\{ \alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n \in \alpha} (F - n) \neq \emptyset \right\} \subseteq \left\{ \alpha \in \mathcal{F}(\mathbb{Z}) : \mu \left(\bigcap_{n \in \alpha} T^{-n}A \right) > 0 \right\},$$

where $\mathcal{F}(\mathbb{Z})$ denote the collection of finite non-empty subsets of \mathbb{Z} . Since P is intersective of order d , there are n_1, \dots, n_d with $FS(\{n_i\}_{i=1}^d) \subset P$ and $a \in F$ with $a + FS(\{n_i\}_{i=1}^d) \subset F$, i.e. $F \cap \bigcap_{n \in FS(\{n_i\}_{i=1}^d)} (F - n) \neq \emptyset$. By (C.1) $P \in \mathcal{F}_{Pd}$.

Now assume that $P \in \mathcal{F}_{P_d}$ and $F \in \mathcal{F}_{pubd}$. Then by the Furstenberg corresponding principle, there are a measure preserving system (X, \mathcal{B}, μ, T) and $A \in \mathcal{B}$ such that $\mu(A) = BD^*(F) > 0$ and

$$(C.2) \quad BD^*\left(\bigcap_{n \in \alpha} (F - n)\right) \geq \mu\left(\bigcap_{n \in \alpha} T^{-n}A\right)$$

for all $\alpha \in \mathcal{F}(\mathbb{Z})$. Since $P \in \mathcal{F}_{P_d}$, there are n_1, \dots, n_d with $FS(\{n_i\}) \subset P$ and $\mu(A \cap \bigcap_{n \in FS(\{q_i\}_{i=1}^d)} T^{-n}A) > 0$. This implies $F \cap \bigcap_{n \in FS(\{n_i\})} (F - n) \neq \emptyset$ by (C.2). \square

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