FORCING RELATION ON PATTERNS OF INVARIANT SETS AND REDUCTIONS OF INTERVAL MAPS

JIE-HUA MAI AND SONG SHAO

ABSTRACT. A pair (X, ψ) is a line system if $X \subset \mathbb{R}$ is nonempty and bounded, $\psi : X \to X$ is continuous, and ψ can be extended to an interval map. Two line systems (X, ψ), (Y, ξ) have the same pattern if there exists an order preserving bijection $h : X \to Y$ such that $h \circ \psi = \xi \circ h$. Say (X, ψ) forces (Y, ξ) if every interval map having an invariant set with the pattern of (X, ψ) also has an invariant set with the pattern of (Y, ξ). Let $J \subset I$ be compact intervals. $g \in C^0(J)$ is called a reduction of $f \in C^0(I)$ if each point $x \in \Delta(f, g) \equiv \{x \in J : g(x) \neq f(x)\}$ is wandering under g and g is constant on every connected component of $\Delta(f, g)$. In this paper we show that for any $f \in C^0(I)$ and any nonempty invariant set S of f there exists a reduction $g \in C^0(J)$ of f with $J = [\inf S, \sup S]$ such that $g|_S = f|_S$ and g is monotonic on every connected component of $J - S$. By means of reductions of maps, we obtain several general results about the forcing relation between the patterns of invariant sets of interval maps, and extend known results about forcing relations between patterns of periodic orbits, also obtaining sufficient conditions for a general pattern to force a given minimal pattern. Moreover, as applications of the idea of reductions of interval maps and forcing relations on patterns, we give a new and simple proof of the converse of Sharkovskiĭ theorem and study fissions of periodic orbits, entropies of patterns etc.

CONTENTS

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1. Introduction

1.1. **Background and preliminaries.** In this paper we study the forcing relation on patterns of line systems and reductions of interval maps. An **interval map** is a continuous map from an interval to itself. A **line system** is a pair (X, ψ), where X is a nonempty bounded subset of $\mathbb{R}$, ψ is a continuous map from X to X, and ψ can be extended to an interval map, that is, there exists an interval map $f : I \to I$ such that $X \subset I$ and $f|_X = \psi$. For any interval map $g : J \to J$ and any nonempty invariant set Y of g (i.e. $g(Y) \subset Y$), the pair $(Y, g|_Y)$ is called a **subsystem** of g. Hence, a pair (X, ψ) is a line system if and only if it is a subsystem of some interval map, and a line system can be regarded as a nonempty invariant set of some interval map. Partitioning all line systems by an "order-preserving" relation (the definition will be given below), we obtain various equivalence classes, which are called patterns of these line systems.

Every interval map has infinitely many subsystems. So there is a natural question: if one has known that an interval map has some subsystem, then what can one say about other subsystems of this map? If one confines his attention to periodic systems, then there are rich results on this question. They are included in the theory of forcing relation on patterns of periodic orbits [1, 5, 9, 24].

One uses $\mathbb{R}$ ($\mathbb{N}$ and $\mathbb{Z}$ respectively) to denote the set of the real numbers (the natural numbers and integers respectively), and denote $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$. Let $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}$ and $\psi : P \to P$. Then (P, ψ) is a **periodic orbit** (or a **cycle**) if ψ ia a cyclic permutation of P. Two periodic orbits (P, ψ), (Q, ξ) are **equivalent** if there exits an order preserving bijection $h : P \to Q$ such that $h \circ \psi = \xi \circ h$, or $h\psi = \xi h$ for short. An equivalence class of this relation will be called a **pattern**. If A is a pattern and $(P, \psi) \in A$, then one says that the cycle (P, ψ) has a pattern A (or P is a **representative** of A) and uses the symbol $[(P, \psi)]$ to denote the pattern A.

There is another equivalent way to define the pattern of periodic orbits. Let (P, ψ) be a periodic orbit and $P = \{p_1 < p_2 < \ldots < p_n\}$. Then the pattern of (P, ψ) is defined to be a cyclic permutation θ of $\mathbb{Z}_n$ which satisfies $\psi(p_i) = p_{\theta(i)}$ for $i = 1, 2, \ldots, n$. It is easy to see that these two definitions are equivalent.

A map $f : I \to I$ has a cycle (P, ψ) if $f|_P = \psi$. One shall say that f **exhibits** the pattern $[(P, \psi)]$. A map f is called a **monotonic extension of** P if it is monotone between consecutive element of P and constant to left of the leftmost and to the right of the rightmost of P. Now one can define forcing relation between patterns: A pattern A **forces** a pattern B if each interval map exhibiting A exhibits also B.

One of important results is that the forcing relation on periodic patterns is a partial order relation [1, 2]. There is also a convenient way to decide the forcing relation on two patterns: let (P, ψ) be a cycle and B a pattern, then $[(P, \psi)]$ forces B if and only if there is a monotonic extension of (P, ψ) which exhibits pattern B [1, 24].

In the theory of discrete dynamical systems, periodic orbits play a very important role. The notion of pattern and forcing relation is the key to the problem of coexistence of various types of cycles for a given map. If we know which patterns are forced by a given pattern A, we have enormous information about the structure of an interval map with cycle of pattern A. Unfortunately, the forcing relation is rather complicated. Therefore it makes sense to consider notations weaker than pattern. This limits the information we get, but makes it easier to obtain it. One of such notions is period. The Sharkovskiĭ Theorem gives the forcing relation among periods. This is a linear ordering, so the characterization of all the periods forced by the given one is simple. Now we state the Sharkovskiĭ Theorem briefly.

Let $I = [a,b]$ be a compact interval and $C^0(I)$ be the set of all continuous maps from I to itself. For any $f \in C^0(I)$ and $x \in I$, denote $\mathcal{O}(x) = \mathcal{O}(x,f) = \{x, f(x), f^2(x), \ldots\}$ and $\mathcal{O}_n(x) = \mathcal{O}_n(x,f) = \{x, f(x), \ldots, f^n(x)\}$. $\mathcal{O}(x,f)$ is called an **orbit** of f. A point $x \in I$ is called a **periodic point of** f **with period** n if $f^n(x) = x$ and $f^k(x) \neq x$ for $1 \leq k < n$. Denote by $P_n(f)$ the set of all periodic points of f with period n and let $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$. Write

$$F_n(I) \equiv F(I,n) = \{f \in C^0(I) : P_n(f) \neq \emptyset\} \tag{1.1}$$

For any $m, n \in \mathbb{N}$, say that m **forces** n and write $m \lhd n$ if $F_m(I) \subset F_n(I)$. In 1964, Sharkovskiĭ discovered the following striking theorem.

Theorem 1.1. $3 \lhd 5 \lhd 7 \lhd \ldots \lhd 6 \lhd 10 \lhd 14 \lhd \ldots \lhd 12 \lhd 20 \lhd 28 \lhd \ldots\ldots \lhd 8 \lhd 4 \lhd 2 \lhd 1.$

The original proof of Theorem 1.1 was given in [25]. Besides this proof, some authors also gave variant proofs, see [11, 16, 20, 22, 26] etc. In most of these proofs, the idea of Straffin [27] concerning directed graphs was adopted.

As a supplement of Theorem 1.1, Sharkovskiĭ [25] also proved the following theorem, which is called the converse of Sharkovskiĭ's theorem by Elaydi [19].

Theorem 1.2. *For any $m, n \in \mathbb{N}$ with $m \neq n$, if $m \lhd n$ then $F_n(I) - F_m(I) \neq \emptyset$. Moreover, let $F(I, 2^\infty) = \bigcap_{k=0}^{\infty} F(I, 2^k)$, $\Phi(I, 2^\infty) = \bigcup\{F_n(I) : n \in \mathbb{N} - \{2^{k-1} : k \in \mathbb{N}\}\}$. Then $F(I, 2^\infty) - \Phi(I, 2^\infty) \neq \emptyset$.*

Unfortunately, the classification of cycles by period only is very coarse. Knowing only periods of cycles is much less than knowing their patterns. Later some other possible choices were discovered ([5, 6, 15]). It gives better classification than just by period, and on the other hand, it admits a full description of possible sets of types. For example, one defines the *rotation pair* of a cycle as (p,q), where q is the period of the cycle and p is the number of its elements which are mapped to the left of themselves. The number p/q ia called the *rotation number* of the cycle. Let $f : I \to I$ be an interval map and let P be a cycle of f of period $q > 1$. Let m be the number of points $x \in P$ such that $f(x) - x$ and $f^2(x) - f(x)$ have different signs. Then the pair $(m/2, q)$ is called the *over-rotation pair* of the cycle. For rotation and over-rotation pairs the equivalence classes are not as large as for period. The difference is that whereas for the rotation pairs forcing is only a partial ordering, for the over-rotation pairs it is a linear one. So using over-rotation pairs one gets a situation that is basically not more complicated than the situation for period, but one digs much deeper into the structure of cycles. We do not plan to discuss these too much in this paper, and please refer to [5, 6, 15] for more details.

In this paper, though we will discuss periodic patterns frequently, our main purpose is to study the forcing relation under more general situations. In [13] the author generalized the forcing relation to minimal piecewise monotone patterns. Let $\mathfrak{M}$ be the set of pairs (X, g) such that $X \subset \mathbb{R}$ is compact, $g \colon X \to X$ is continuous, g is minimal on X and has a piecewise monotonic extension to the convex hull of

X. Two pairs $(X,g),(Y,f)$ from $\mathfrak{M}$ are equivalent if the map $h\colon \mathcal{O}(\min X,g) \to \mathcal{O}(\min Y,f)$, defined by $h(g^m(\min X)) = f^m(\min Y)$ for each $m \geq 0$, is increasing on $\mathcal{O}(\min X,g)$. An equivalence class of this relation is defined to be a minimal pattern. In [13] the author showed that the forcing relation on minimal piecewise monotone patterns is a partial ordering.

1.2. **Main results of the paper.** In this paper we study forcing relation on patterns of invariant sets of interval maps, and our viewpoint is different from [13] (also we do not assume that the maps considered are piecewise monotone).

Let Ψ be the set of all line systems (X,ψ). Let $(X,\psi),(Y,\xi) \in \Psi$. Say that (X,ψ) and (Y,ξ) have **the same pattern** (denoted by $(X,\psi) \approx (Y,\xi)$) if there exists an order-preserving bijection $h : X \to Y$ such that $h\psi = \xi h$. Denote by Ψ^* the set of equivalence classes in Ψ under the equivalence relation $\approx$, i.e. $\Psi^* = \Psi/\approx$. Then Ψ^* can be regarded as the set of patterns of invariant sets of interval maps. Unlike patterns of periodic orbit and minimal sets, the forcing relation on Ψ^* is not a partial ordering any more (see Example 4.5, 4.7).

Let $A,B \in \Psi^*$ be two patterns. Say A **forces** B if every interval map having an invariant set with the pattern A also has an invariant set with the pattern B. Our aim is to give some conditions under which A can force B. For this purpose, we develop a tool named the reduction of continuous maps. Let J,I be compact intervals with $J \subset I$, $f \in C^0(I)$ and $g \in C^0(J)$. Write $\Delta(f,g) = \{x \in J : g(x) \neq f(x)\}$. g is called a **reduction of** f if each point $x \in \Delta(f,g)$ is wandering under g and g is constant on every connected component of $\Delta(f,g)$. About the reduction we have the following result:

Theorem 2.8 *For any $f \in C^0(I)$ and any nonempty compact invariant set S of f there exists a reduction $g \in C^0(J)$ of f with $J = [\inf S, \sup S]$ such that $g|_S = f|_S$ and g is monotonic on every connected component of $J - S$.*

By means of reductions of maps, we obtain several general results about the forcing relation between the patterns of invariant sets on intervals. For example, we give a characterization of the forcing relation between patterns in the absence of companionate orbits (see Definition 4.13). We have the following result:

Theorem 4.14 *Let $X \subset \mathbb{R}$ be compact and $(X,\psi),(Z,\xi) \in \Psi$. Suppose ξ has no companionate orbits. Then $[(X,\psi)]$ forces $[(Z,\xi)]$ if and only if there exists a monotonic extension of (X,ψ) which exhibits ξ.*

And for some special patterns, we can weaken the conditions. For example, for the case of periodic patterns, we have

Theorem 5.2 *Suppose $X \subset \mathbb{R}$ is compact, $(X,\psi) \in \Psi$. Then $[(X,\psi)]$ forces a periodic pattern $[(P,\theta)]$ if and only if there exists a monotonic extension of ψ which has a periodic orbit of pattern θ.*

As for non-periodic minimal patterns, we have a sufficient condition for a general pattern to force a given minimal pattern:

Theorem 5.8 *Let (W,φ) be a compact line system, and (X,ψ) be a minimal line system but not periodic. If there exists a monotonic extension f of (W,φ) exhibiting (X,ψ), then each interval map exhibiting (W,φ) has a minimal subset which is equivalent to (X,ψ) in sense of Bobok.*

As applications of the idea of reductions of interval maps, we study periodic patterns and give a new approach to Theorem 1.2. Also we apply the results on forcing relation we built to study fissions of periodic orbits and the entropy of patterns etc. For example, we define $h^*(X,\psi) = \inf\{h(I,f) : f \in C^0(I) \text{ and } f|_X = \psi\}$, where X is compact and $I = [\inf X, \sup X]$. We show that if $(X,\psi) \approx (Y,\xi)$ with X, Y compact, then $h^*(X,\psi) = h^*(Y,\xi)$ (Corollary 7.4). Hence we can define the topological entropy of a pattern.

1.3. **Organization of the paper.** The paper is organized as follows: In Section 2 we introduce the definition of reductions of interval maps and give some basic properties and results. In particular, we will prove Theorem 2.8. In Section 3 as applications of the tool developed in Section 2 we give some results on patterns of periodic orbits. Especially, we give a new and simple proof of Theorem 1.2. Section 4 and 5 are the bulk of the paper, where we study the forcing relation on invariant sets carefully. In Section 4 we give some general conditions under which one pattern can force another one and in Section 5 we discuss the periodic and non-periodic minimal patterns. In Section 6 we use the results developed in Section 4 to study the fissions of periodic orbits etc. Finally, we study the entropy of patterns in the last section.

2. Reductions of interval maps

In this section we introduce the concept of reductions of interval maps, and show that for any $f \in C^0(I)$ and any nonempty invariant set S of f, there exists a reduction of f preserving S.

2.1. **Notation.** For any $\{r,s\} \subset \mathbb{R}$ with $r < s$, write $[r;s] = [s;r] = [r,s]$, and $[r;r] = \{r\}$. For any $\{a,b\} \subset \mathbb{R}$, write $(a;b] = [b;a) = [a;b] - \{a\}$, and $(a;b) = (b;a) = (a;b] - \{b\}$. For any nonempty bounded set $X \subset \mathbb{R}$, let $L[X]$ denote the closed convex hull of X in $\mathbb{R}$, i.e.

$$L[X] = [\inf X,\ \sup X],$$

and let

$$l(X) = \sup X - \inf X.$$

The interior, closure and boundary of X in $\mathbb{R}$ are denoted by $\overset{\circ}{X}$, $\overline{X}$ and ∂X respectively.

Let X be a topological space and $f : X \to X$ be a continuous map. A point $x \in X$ is called a **recurrent point** of f if for any neighborhood U of x and any $m \in \mathbb{N}$ there exists $n > m$ such that $f^n(x) \in U$. A point $x \in X$ is **nonwandering** if for every neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$. Denote respectively by $R(f)$ and $\Omega(f)$ the set of all recurrent and nonwandering points of f. It is clear that both $R(f)$ and $\Omega(f)$ are invariant sets of f.

2.2. **Reductions of interval maps.** Let $I = [a,b]$ and $f \in C^0(I)$. A closed interval $J \subset I$ is called a **level interval** of f if $f|_J$ is constant.

Definition 2.1. Let I, J be compact intervals with $J \subset I$ and $f \in C^0(I)$, $g \in C^0(J)$. Set

$$\Delta(f,g) = \{y \in J : g(y) \neq f(y)\}. \tag{2.1}$$

g is called a **reduction** of f if the following conditions are satisfied:

(a) $\Delta(f,g) \cap \Omega(g) = \emptyset$;

(b) g is constant on every connected component of $\Delta(f,g)$.

It is easy to see that $\Delta(f,g)$ is an open set relative to the topology of J. And if $\Delta(f,g)\cap\partial J=\emptyset$, then every connected component of $\Delta(f,g)$ is an open interval. Note that in Definition 2.1 we do not insist $J\neq I$ or $g\neq f$. Thus f itself is also a reduction of f.

We now exhibit some basic properties on reductions of interval maps.

Lemma 2.2. *Let $J\subset I$ be intervals and $g\in C^0(J)$ be a reduction of $f\in C^0(I)$. Then*
(i) $g|_{\Omega(g)}=f|_{\Omega(g)}$;
(ii) $R(g)\subset R(f)$, *and* $P_n(g)\subset P_n(f)$ *for all* $n\in\mathbb{N}$.
(iii) *For any interval* $K\subset J$, *if* $f|_K$ *is increasing (decreasing, constant respectively), then* $g|_K$ *is increasing (decreasing, constant respectively);*
(iv) $l(g(K))\le l(f(K))$, *for any interval* $K\subset J$;
(v) $\Omega(g)\subset\Omega(f)$.

Proof. It is easy to verify (i)-(iv) by the definition. Now we show (v). Consider any $x\in\Omega(g)$. For any ε-neighborhood U of x in J, there exist $w\in U$ and $m\in\mathbb{N}$ such that $g^m(w)\in U$. If $\{w,g(w),\dots,g^{m-1}(w)\}\cap\Delta(f,g)\neq\emptyset$, then there exists an $n\in\{0,1,\dots,m-1\}$ such that $g^n(w)\in\Delta(f,g)$ and $g^j(w)\notin\Delta(f,g)$ for all $j\in\{n+1,\dots,m-1\}$. Let K_n be the connected component of $\Delta(f,g)$ containing $g^n(w)$. Then $g|_{\overline{K}_n}$ is constant. By (a) of Definition 2.1, $g^n(x)\notin K_n$. Hence there is a $u_n\in[x;w)$ such that $g^n(u_n)\in\overline{K}_n-K_n\subset J-\Delta(f,g)$, $g^j(u_n)=g^{j-n}(g^n(u_n))=g^{j-n}(g^n(w))=g^j(w)\notin\Delta(f,g)$ for $j\in\{n+1,\dots,m-1\}$, and $g^m(u_n)=g^m(w)\in U$. If it still holds that $\{u_n,g(u_n),\dots,g^{n-1}(u_n)\}\cap\Delta(f,g)\neq\emptyset$, then one can also find a $k\in\{0,1,\dots,n-1\}$ and a $u_k\in[x;u_n)$ such that $\{g^j(u_k):j=k,k+1,\dots,m-1\}\cap\Delta(f,g)=\emptyset$ and $g^m(u_k)=g^m(w)\in U$. Thus there must exist a $u=u_0\in[x;w]$ such that $\{u,g(u),\dots,g^{m-1}(u)\}\cap\Delta(f,g)=\emptyset$ and $g^m(u)=g^m(w)\in U$. By (2.1), one has $f^m(u)=g^m(u)\in U$. This implies $x\in\Omega(f)$, and hence $\Omega(g)\subset\Omega(f)$. □

Lemma 2.3. *Let g be a reduction of f, and h be a reduction of g. Then h is a reduction of f.*

Proof. Note that $\Delta(f,h)\subset\Delta(f,g)\cup\Delta(g,h)$. By Definition 2.1-(a) and Lemma 2.2-(v), $\Delta(f,h)\cap\Omega(h)\subset(\Delta(f,g)\cup\Delta(g,h))\cap\Omega(h)\subset(\Delta(f,g)\cap\Omega(g))\cup(\Delta(g,h)\cap\Omega(h))=\emptyset$. by Definition 2.1-(b) and Lemma 2.2-(iii), h is constant on every connected component of $\Delta(g,h)$ and of $\Delta(f,g)$. Thus h is a reduction of f. □

Proposition 2.4. *Let $I_0\supset I_1\supset I_2\supset\dots$ and $J=\bigcap_{n=0}^{\infty}I_n$ be compact intervals and $f_{n+1}\in C^0(I_{n+1})$ be a reduction of $f_n\in C^0(I_n)$ for $n=0,1,2,\cdots$. Then $f_0|_J,f_1|_J,f_2|_J,\dots$ converges uniformly to a map $g\in C^0(J)$, which is also a reduction of f_0.*

Proof. By Lemma 2.3, each f_n is a reduction of f_0. Suppose $J=[c,d]$. Let $P_1=\bigcap_{n=0}^{\infty}P_1(f_n)$. For $n\ge 0,P_1(f_n)$ is a nonempty closed set. By Lemma 2.2-(ii), $P_1(f_{n+1})\subset P_1(f_n)$. Thus P_1 is nonempty. By Lemma 2.2-(i), $f_0|_{P_1}=f_1|_{P_1}=f_2|_{P_1}=\dots$. Take a point $e\in P_1$. For any $x\in[c,e]$ (resp. $x\in[e,d]$), let $Y_{x,n}$ be the connected component of $f_n^{-1}(f_n(x))\cap[c,e]$ (resp. $f_n^{-1}(f_n(x))\cap[e,d]$) containing x, and let $y_{x,n}=\max Y_{x,n}$ (resp. $y_{x,n}=\min Y_{x,n}$). It follows from (b) of Definition

2.1 that $y_{x,n} \notin \Delta(f_0, f_n)$. Hence $f_n(x) = f_n(y_{x,n}) = f_0(y_{x,n})$. By Lemma 2.2-(iii), $Y_{x,0} \subset Y_{x,1} \subset Y_{x,2} \subset \dots$. Thus $y_{x,0}, y_{x,1}, y_{x,2}, \dots$ is a monotonic sequence. Let $y_x = \lim_{n\to\infty} y_{x,n}$. Then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_0(y_{x,n}) = f_0(y_x)$. Define $g : J \to J$ by

$$g(x) = f_0(y_x), \quad \text{for any} \quad x \in J.$$

Then $f_0|_J, f_1|_J, f_2|_J, \dots$ converges pointwisely to g. For any closed interval $K \subset J$, it follows from Lemma 2.2-(iv) that

$$l(f_0(K)) \geq l(f_1(K)) \geq l(f_2(K)) \geq \dots.$$

Thus $\{f_n|_J : n = 0, 1, 2, \dots\}$ is an equicontinuous family of maps, and hence $f_0|_J$, $f_1|_J$, $f_2|_J$, $\dots$ converges uniformly to g. This implies that g is continuous.

For any $x \in \Delta(f_0, g)$, there exists an $m \geq 1$ such that $x \in \Delta(f_0, f_m)$. By Definition 2.1, there is an open ε-neighborhood U_x of x in J such that $f_m(U_x) = \{f_m(x)\}$ and $U_x \cap \Omega(f_m) = \emptyset$. For all $n \geq m$, by (iii) and (v) of Lemma 2.2 one gets $f_n(U_x) = \{f_n(x)\}$ and $U_x \cap \Omega(f_n) = \emptyset$, which imply that $g(U_x) = \{g(x)\}$ and $\mathcal{O}(f_n(x), f_n) \cap U_x = \emptyset$. Since f_n converges uniformly to g as $n \to \infty$, one has $\mathcal{O}(g(x), g) \cap U_x = \emptyset$, which imply $x \notin \Omega(g)$. Thus g satisfies the conditions (a) and (b) in Definition 2.1 for $f = f_0$, and hence g is a reduction of f_0. This completes the proof of Proposition 2.4. □

2.3. **Normal reduction of f preserving S.** Now we are going to show that for any $f \in C^0(I)$ and invariant set S of f, there exists a reduction of f preserving S. Actually we can say more.

Definition 2.5. Let $f \in C^0(I)$, and S be a nonempty invariant set of f. A map $g \in C^0(L(S))$ is called a **normal reduction of f preserving** S if g is a reduction of f, $g|_S = f|_S$, and g is monotonic on every connected component of $L(S) - S$.

Remark 2.6. Note that, for any $g \in C^0(L(S))$ and any $x \in L(S)$, $g|_{\{x\}}$ is monotonic. Thus, g is monotonic on every connected component of $L(S) - S$ if and only if g is monotonic on every connected component of $L(S) - \overline{S}$, and hence, g is a normal reduction of f preserving S if and only if g is a normal reduction of f preserving $\overline{S}$.

Example 2.7. Let $f : [0, 1] \to [0, 1]$ be a piecewise linear map whose graph is in Figure 1. It is clear that $S = \{0, 1/2, 1\}$ is an invariant set of f. Then as in Figure 1 g is a normal reduction of f preserving S.

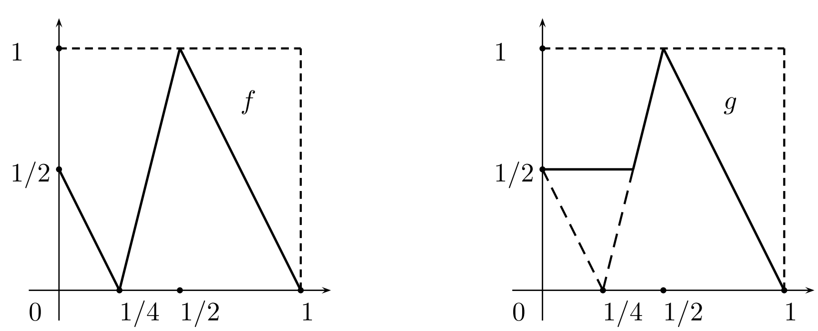

Figure 1

The main result of this part is the following:

Theorem 2.8. *Let $f \in C^0(I)$ and S be a nonempty invariant set of f. Then there exists a normal reduction of f preserving S.*

To prove Theorem 2.8, one need some preparations.

Definition 2.9. Let $I = [a, b]$, $f \in C^0(I)$ and S be a nonempty invariant set of f. An interval $J = [v, y]$ is called a **pseudo-levelable interval of** (S, f) if $J \subset L(S), \overset{\circ}{J} \cap S = \emptyset, f(v) = f(y)$ and $l(f(J)) > 0$. A pseudo-levelable interval $J = [v, y]$ of (S, f) is said to be **levelable** if $\mathcal{O}(v) \cap \overset{\circ}{J} = \emptyset$.

Write

$$\lambda(S, f) = \sup(\{0\} \cup \{l(f(J)) : J \text{ is a levelable interval of } (S, f)\}), \tag{2.2}$$

$$\mu(S, f) = \sup(\{0\} \cup \{l(f(J)) : J \text{ is a pseudo-levelable interval of } (S, f)\}). \tag{2.3}$$

A levelable interval J of (S, f) is said to be **maximal** if $l(f(J)) = \lambda(S, f)$ and $l(J) = \max\{l(K) : K \text{ is a levelable interval of } (S, f) \text{ with } l(f(K)) = \lambda(S, f)\}$. Similarly, one can define maximal pseudo-levelable intervals of (S, f).

Obviously, if $\lambda(S, f) > 0$ (resp. $\mu(S, f) > 0$), then there exists a maximal levelable (resp. maximal pseudo-levelable) interval of (S, f).

Remark 2.10. From Definition 2.9 we see that, for any $f \in C^0(I)$ and any nonempty invariant set S of f, an interval $J \subset L(S)$ is a levelable (resp. pseudo-levelable) interval of (S, f) if and only if J is a levelable (resp. pseudo-levelable) interval of $(\overline{S}, f)$. Hence, we have $\lambda(S, f) = \lambda(\overline{S}, f)$ and $\mu(S, f) = \mu(\overline{S}, f)$.

One can get the following two lemmas readily.

Lemma 2.11. *Let S be a nonempty invariant set of $f \in C^0(I)$. Then $\mu(S, f) = 0$ if and only if f is monotonic on every connected component of $L(S) - S$.*

Lemma 2.12. *Let S be a nonempty invariant set of $f \in C^0(I)$, and $J = [v, y]$ be a levelable interval of (S, f). Define $\varphi : I \to I$ by*

$$\varphi(x) = \begin{cases} f(x), & \text{if } x \in I - J; \\ f(v), & \text{if } x \in J. \end{cases}$$

Then φ is a reduction of f, and $\overset{\circ}{J} \cap \Omega(\varphi) = \emptyset$. (See Figure 2)

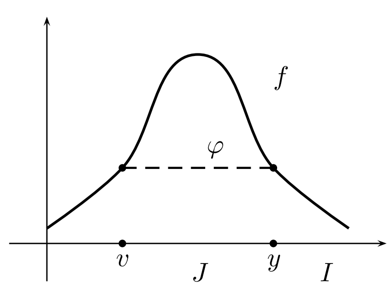

Figure 2

Definition 2.13. The map $\varphi : I \to I$ defined in Lemma 2.12 is called a **basic reduction of f by leveling J**.

Lemma 2.14. *Let $J \subset I$ be compact intervals, $f \in C^0(I)$ and $g \in C^0(J)$. Suppose S is a nonempty invariant set of g and $g|_S = f|_S$. If g is a reduction of f, then $\mu(S,g) \le \mu(S,f)$.*

Proof. Assume $K = [v,y]$ is a pseudo-levelable interval of (S,g). Then $g(v) = g(y)$. Let K_v (resp. K_y) be the connected component of $g^{-1}(g(v))$ containing v (resp. y). Suppose $\max K_v = v_1$, $\min K_y = y_1$. Then by Definition 2.1-(b), one has $f(v_1) = g(v_1) = g(v) = g(y) = g(y_1) = f(y_1)$, and by Lemma 2.2-(iv) one has $l(g([v,y])) = l(g([v_1,y_1])) \le l(f([v_1,y_1]))$. Thus $[v_1,y_1]$ is a pseudo-levelable interval of (S,f), and from (2.3) it follows that $\mu(S,g) \le \mu(S,f)$. □

Lemma 2.15. *Let S be a nonempty invariant set of $f \in C^0(I)$. Then*

$$\mu(S,f) \ge \lambda(S,f) \ge \mu(S,f)/4.$$

Proof. It is clear that $\mu(S,f) \ge \lambda(S,f)$. It is left to show that $\lambda(S,f) \ge \mu(S,f)/4$. Since $\mu(S,f) = 0$ implies $\lambda(S,f) = 0$, one may assume that $\mu(S,f) > 0$. Let $J = [v;y]$ be a pseudo-levelable interval of (S,f) satisfying $l(f(J)) = \mu(S,f)$. Take u and $u_1 \in J$ such that $f(u) = \max f(J)$ and $f(u_1) = \min f(J)$. Then $f(u) - f(u_1) = l(f(J))$. By symmetry, one may assume $f(u) - f(y) \ge l(f(J))/2$. Let $(z;z_1)$ be the connected component of $L(S) - S$ containing $\mathring{J}$. One may assume $f(z_1) \ge f(z)$ and $v \in [z;y)$.

If $f(u) \ge f(z_1)$, then there exists $w \in [z;u]$ such that $f(w) = f(z_1)$. Since $[w;z_1]$ is a levelable interval of (S,f), $\lambda(S,f) \ge l(f([w;z_1])) \ge f(u) - f(y) \ge l(f(J))/2 = \mu(S,f)/2$.

If $f(y) \le f(z)$, then there exists $z_2 \in [y;z_1]$ such that $f(z_2) = f(z)$. Since $[z;z_2]$ is a levelable interval of (S,f), $\lambda(S,f) \ge l(f([z;z_2])) \ge l(f(J)) = \mu(S,f)$.

If $f(u_1) \le f(z) < f(y)$ and $u_1 \in (u;y)$, then there exists $z_3 \in [u_1;y)$ such that $f(z_3) = f(z)$ and one also has $\lambda(S,f) \ge l(f([z;z_3])) \ge f(u) - f(u_1) = \mu(S,f)$.

In the following we assume

$$f(z) < f(v) = f(y) < f(u) < f(z_1)$$

and

$$u_1 \in [v;u) \quad \text{or} \quad u_1 \in (u;y] \ \text{ and } \ f(u_1) > f(z).$$

Then there exist $w_1 \in (y;z_1), y_1 \in (u;w_1)$ and $v_1 \in (z;u)$ such that

$$f(w_1) = f(u), \quad f(y_1) = f(v_1) = \min(f([u;w_1]))$$

and

$$f([v_1;u]) = f([u;y_1]) = f([y_1;w_1]) = [f(y_1), f(u)].$$

Note that we have

$$z < v_1 < u < y_1 < w_1 < z_1 \ \text{ or } \ z > v_1 > u > y_1 > w_1 > z_1.$$

If $\mathcal{O}(u) \cap (u;w_1) = \emptyset$, then $[u;w_1]$ is a levelable interval of (S,f) and $\lambda(S,f) \ge l(f([u;w_1])) \ge f(u) - f(y) \ge \mu(S,f)/2$. If $\mathcal{O}(u) \cap (u;w_1) \ne \emptyset$, then there exist $j \in \mathbb{N}$ such that $f^j(u) = f^j(w_1) \in (u;w_1)$. This implies $P(f) \cap [u;w_1] \supset P_1(f^j) \cap (u;w_1) \ne \emptyset$. For any $x \in P(f)$, let $p(x)$ be the period of x under f. Take $x_0 \in P(f) \cap [v_1;w_1]$ such that

$$p(x_0) = \min\{p(x) : x \in P(f) \cap [v_1;w_1]\}. \tag{2.4}$$

Suppose $p(x_0) = k$. Then $k \le j$. Take $x_1 \in [v_1; u]$ and $x_2 \in [y_1; w_1]$ such that $f(x_1) = f(x_2) = f(x_0)$ and $f^{-1}(f(x_0)) \cap [v_1; w_1] \subset [x_1; x_2]$. Then $(\mathcal{O}(x_0) - \{x_0\}) \cap [x_1; x_2] = \emptyset$ since otherwise there would be a point $y_0 \in P(f) \cap (x_1; x_2)$ with $p(y_0) \le k - 1$, which contradicts (2.4).

If $x_0 \in [v_1; u]$, then $[x_0; x_2]$ is a levelable interval of (S, f) and one has $\lambda(S, f) \ge l(f([x_0; x_2])) = f(u) - f(y_1) \ge f(u) - f(y) \ge \mu(S, f)/2$. If $x_0 \in [y_1; w_1]$, then one also has $\lambda(S, f) \ge l(f([x_1; x_0])) \ge f(u) - f(y_1) \ge \mu(S, f)/2$. If $x_0 \in [u; y_1]$, then one still has

$$\begin{cases} \lambda(S, f) \ge l(f([x_0; x_2])) \ge f(x_0) - f(y_1) \ge \mu(S, f)/4, & \text{if } f(x_0) \ge [f(u) + f(y_1)]/2; \\ \lambda(S, f) \ge l(f([x_1; x_0])) \ge f(u) - f(x_0) \ge \mu(S, f)/4, & \text{if } f(x_0) \le [f(u) + f(y_1)]/2. \end{cases}$$

The proof of Lemma 2.15 is completed. □

Now it is time to prove Theorem 2.8.

Proof of Theorem 2.8 Firstly, let $f_0 = f$. For $k \ge 0$, suppose $f_k \in C^0(I)$ has been defined. If $\lambda(S, f_k) = 0$, then let $J_k = \emptyset$ and $f_{k+1} = f_k$. If $\lambda(S, f_k) > 0$, then take a maximal levelable interval $J_k = [v_k, y_k]$ of (S, f_k) and let $f_{k+1} \in C^0(I)$ be the basic reduction of f_k by leveling J_k. Continuing this process, one has sequences $\{f_n\}_{n=0}^{\infty}$ and $\{J_n\}_{n=0}^{\infty}$.

Claim. $\lim_{k\to\infty} \lambda(S, f_k) = 0$.

Proof of Claim. If Claim does not hold, then there exist $\varepsilon > 0$ and infinitely many positive integers $k_1 < k_2 < k_3 < \dots$ such that

$$l(f_{k_n}(J_{k_n})) = \lambda(S, f_{k_n}) \ge \varepsilon \quad \text{for all} \quad n \ge 1, \tag{2.5}$$

$$\lim_{n\to\infty} v_{k_n} = w \quad \text{and} \quad \lim_{n\to\infty} y_{k_n} = z \quad \text{for some } w, z \in L(S). \tag{2.6}$$

Since f is uniformly continuous, there is $\delta = \delta(\varepsilon) > 0$ such that

$$l(f(J)) < \varepsilon/3 \quad \text{for every interval} \quad J \subset I \quad \text{with} \quad l(J) < \delta.$$

By (2.6), there exists $m > 1$ such that $|v_{k_{m+1}} - v_{k_m}| < \delta$ and $|y_{k_{m+1}} - y_{k_m}| < \delta$. According to Lemma 2.3, Lemma 2.2-(iv) and (2.6) one has that

$$\begin{aligned} l(f_{k_{m+1}}(J_{k_{m+1}})) &\le l(f([v_{k_{m+1}}; v_{k_m}])) + l(f_{k_{m+1}}(J_{k_m})) + l(f([y_{k_m}; y_{k_{m+1}}])) \\ &< \varepsilon/3 + 0 + \varepsilon/3 < \varepsilon. \end{aligned}$$

But this contradicts (2.5). This completes the proof of Claim.

By above claim and Lemma 2.15, one has $\lim_{n\to\infty} \mu(S, f_k) = 0$. By Proposition 2.4, $f_0, f_1, f_2, \dots$ converges uniformly to a map $g \in C^0(I)$, which is a reduction of f_n for all $n \ge 0$. Clearly, S is still an invariant set of g, and $g|_S = f|_S$. By Lemma 2.14 one has $\mu(S, g) = 0$. Put $\varphi = g|_{L(S)}$. Then $\varphi \in C^0(L(S))$ is a reduction of f, $\varphi|_S = g|_S = f|_S$, and $\mu(S, \varphi) = \mu(S, g) = 0$. By Lemma 2.6, φ is monotonic on every connected component of $L(S) - S$. Thus φ is a normal reduction of f preserving S. The proof of Theorem 2.8 is completed.

3. Some results on periodic patterns

As applications of the idea of reductions of interval maps, in this section we study patterns of periodic orbits. Especially, we will give a new and simple proof of the converse of Sharkovskiĭ's Theorem (Theorem 1.2). Usually, the proof of Theorem 1.2 is gotten by constructing some concrete examples (see [26, 18]). But here we use the method of reductions of maps.

3.1. **Relation $\lhd$ on $\mathbb{N}$.** Firstly recall some notations. Let $I = [a, b]$ and $F_n(I)$ be the same as in (1.1). For any $m, n \in \mathbb{N}$ with $m \neq n$, write $m \lhd n$ or $n \rhd m$ if $F_m(I) \subset F_n(I)$. Then one obtains a relation $\lhd$ on $\mathbb{N}$. Write

$$F_n^*(I) \equiv F^*(I, n) = F_n(I) - \bigcup\{F_m(I) : m \in \mathbb{N} \text{ and } m \lhd n\}.$$

For any finite set T, let $|T|$ denote the cardinality of T.

The main results of this part is Theorem 3.4 and Theorem 3.5. To prove them we need some notions and lemmas.

Definition 3.1. Let $f \in C^0(I)$ and $X = \{x_1 < x_2 < \ldots < x_n\}$ be a periodic orbit of f with period $n \geq 1$. X is said to be **in an odd** (resp. **even**) **state** under f if for each $i \in \mathbb{Z}_n$ there exists an open interval J_i with $x_i \in J_i \subset I$ such that $f|_{J_i}$ is monotonic but not constant and the cardinality $|\{i \in \mathbb{Z}_n : f|_{J_i} \text{ is decreasing }\}|$ is odd (resp. even).

Lemma 3.2. *Let $f \in C^0(I)$ and $x \in P_n(f), n \geq 1$. Suppose $\mathcal{O}(x)$ is in an odd or even state under f. Then for any given integer $k \geq 2$, there exists an open interval $U = U_k$ with $x \in U \subset I$ such that*

(i) *$U \cap P_i(f) = \emptyset$ for any $i \in \{1, 2, \ldots, kn\} - \{n, 2n\}$;*

(ii) *If there exists $y \in U \cap P_n(f) - \{x\}$, then $\mathcal{O}(x)$ is in an even state under f, and $[x; y] \cap \Omega(f) - P_n(f) = \emptyset$;*

(iii) *If there exists $y \in U \cap P_{2n}(f)$, then $\mathcal{O}(x)$ is in an odd state under $f, x \in (y; f^n(y))$, and $[y; f^n(y)] \cap \Omega(f) - \{x\} - P_{2n}(f) = \emptyset$.*

Proof. Suppose $\mathcal{O}(x) = \{x_1 < x_2 < \ldots < x_n\}$. Let $J_1, J_2, \ldots, J_n$ be as in Definition 3.1. Then there exists an open interval $U = U_k$ with $x \in U \subset I$ such that $\bigcup_{j=0}^{kn+n} f^j(U) \subset \bigcup_{i=1}^{n} J_i$, and $(\bigcup_{m=0}^{k} f^{mn+i}(U)) \bigcap (\bigcup_{m=0}^{k} f^{mn+j}(U)) = \emptyset$ for $0 \leq i < j < n$. Note that $f^n|_U$ is increasing (resp. decreasing) if $\mathcal{O}(x)$ is in an even (resp. odd) state under f. It is easy to check that the properties (i)$-$(iii) hold. □

Lemma 3.3. *If $f \in C^0(I)$ has a periodic orbit Q of period m, then for any $n \rhd m$, f has a periodic orbit of period n contained in $L(Q)$.*

Proof. Let g be a normal reduction of f preserving Q. Since $m \lhd n$, g has a periodic orbit Q' of period n. Since $g \in C^0(L(Q))$, one has $Q' \subset L(Q)$. Since g is a reduction of f, Q' is also a periodic orbit of f. □

Theorem 3.4. *For any $f \in C^0(I)$ and $n \in \mathbb{N}$, if $P_n(f) \neq \emptyset$ then there exist a periodic orbit Q of f with period n and a normal reduction φ of f preserving Q such that $P_m(\varphi) = \emptyset$ for all $m \lhd n$.*

Proof. Let Q_0 be a periodic orbit of f with period n, and let g be a normal reduction of f preserving Q_0. Then g is piecewise monotonic. Let $S_n(g) = \{x \in P_n(g) : x = \min \mathcal{O}(x, g)\}$ and let $v = \sup S_n(g)$. If $v \in P_n(g)$, take $Q = \mathcal{O}(v, g)(= \mathcal{O}(v, f))$ and let φ be a normal reduction of g preserving Q. Then $P_m(\varphi) = \emptyset$ for all $m \lhd n$. If not, it follows from Lemma 3.3 that g has a periodic orbit of period n contained in $L(Q)$, which contradicts the definition of v.

If $v \notin P_n(g)$, then $v \in P_k(g)$ for some divisor k of n. By Lemma 3.2, $\mathcal{O}(v, g)$ must be in an odd state under g and $k = n/2$. Choose $y \in S_n(g)$ such that $v - y$ is sufficiently small. Then $[y, v) \cap \Omega(g) \subset P_n(g)$. Take $Q = \mathcal{O}(y, g)$ and let φ be a normal reduction of g preserving Q. Then we still have $P_m(\varphi) = \emptyset$ for all $m \lhd n$. If not, φ has a periodic orbit of period m contained in $L(Q)$. Since $[y, v) \cap \Omega(g) \subset P_n(g)$, this m periodic orbit is contained in $L(Q) - [y, v]$. Then by Lemma 3.3 φ (and hence g) has a periodic orbit of period n contained in $L(Q)-[y, v]$, which contradicts the definition of v. $\square$

From Theorem 3.4 we see that $F_m(I) \neq F_n(I)$ for all $m \neq n \in \mathbb{N}$. Moreover we have:

Theorem 3.5. *Let $m_1, m_2, m_3, \ldots$ and $n_1, n_2, n_3, \ldots$ be two infinite sequences of positive integers. Suppose $m_i \lhd n_j$ for all $i, j \in \mathbb{N}$. Then*

$$\bigcap_{i=1}^{\infty} F_{n_i}(I) - \bigcup_{j=1}^{\infty} F_{m_j}(I) \neq \emptyset.$$

Proof. Let $a_0 = a, a_j = (a_{j-1} + b)/2$, and $I_j = [a_{j-1}, a_j]$ for $j = 1, 2, \ldots$. Then by Theorem 3.4 one can construct a map $g \in C^0(I)$ such that $g(I_j) \subset I_j$ and $g|_{I_j} \in F^*_{n_j}(I_j)$ for all $j \geq 1$. Clearly, $g \in \bigcap_{i=1}^{\infty} F_{n_i}(I) - \bigcup_{j=1}^{\infty} F_{m_j}(I)$. $\square$

3.2. **Patterns of periodic orbits.** Now we study patterns of periodic orbits. Denote $\mathbf{C}_n$ the set of all cyclic permutations of $\mathbb{Z}_n$ and

$$\mathbf{C} = \bigcup_{n=1}^{\infty} \mathbf{C}_n.$$

Let $I = [a, b]$, $f \in C^0(I)$ and $Q = \{x_1 < x_2 < \ldots < x_n\}$ be a periodic orbit of f. A cyclic permutation $\theta \in \mathbf{C}_n$ is called a **pattern** of Q if $f(x_i) = x_{\theta(i)}$ for any $i \in \mathbb{Z}_n$. Let

$$F_\theta(I) \equiv F(I, \theta) = \{f \in C^0(I) : f \text{ has a periodic orbit of pattern } \theta\}.$$

For any γ and $\theta \in \mathbf{C}$ with $\gamma \neq \theta$, one says that γ **forces** θ if $F_\gamma(I) \subset F_\theta(I)$, and denote it by $\gamma \to \theta$ or $\theta \leftarrow \gamma$. Then one obtains a transitive relation $\to$ on $\mathbf{C}$. The relation $\to$ is a refinement of the Sharkovskiĭ ordering $\lhd$, which has been studied by lots of authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 23, 24] etc.). In the following we will also study this relation. By means of reductions of maps, we can get some new results.

For any $\theta \in \mathbf{C}$ and any $\Theta \subset \mathbf{C}$, let

$$\mathbf{E}(\theta) = \{\gamma : \gamma \in \mathbf{C} \text{ and } \gamma \to \theta\}, \quad \mathbf{E}(\Theta) = \{\gamma \in \mathbf{C} : \gamma \to \theta \text{ for all } \theta \in \Theta\}.$$

and

$$F^*_\theta(I) \equiv F^*(I, \theta) = F_\theta(I) - \bigcup\{F_\gamma(I) : \gamma \in \mathbf{C} \text{ and } \gamma \to \theta\}.$$

With a few changes in the proofs of Theorems 3.4 and 3.5 (for example, changing "periodic orbit of period n " to "periodic orbit of pattern θ " etc.), one has the following two theorems.

Theorem 3.6. *For any $f \in F_\theta(I)$, there exist a periodic orbit Q of f with pattern θ and a normal reduction φ of f preserving Q such that*

$$\varphi \in F^*_\theta(L(Q)) = F_\theta(L(Q)) - \bigcup\{F_\gamma(L(Q)) : \gamma \in \mathbf{E}(\theta)\}.$$

Theorem 3.7. *For any $\Theta \subset \mathbf{C}$ with $\Theta \neq \emptyset$,*

$$\bigcap\{F_\theta(I) : \theta \in \Theta\} - \bigcup\{F_\gamma(I) : \gamma \in \mathbf{E}(\Theta)\} \neq \emptyset.$$

As a corollary of Theorem 3.6, one has the following proposition, which is due to Baldwin [2].

Proposition 3.8. *Let θ, $\gamma \in \mathbf{C}$ with $\theta \neq \gamma$. If $\gamma \to \theta$, then $\theta \not\to \gamma$.*

Definition 3.9. Let $\gamma \in \mathbf{C}_n, n \geq 1$, and $\eta \in \mathbf{C}_{2n}$. η is called a **doubling** of γ if $\eta(\{2i-1, 2i\}) = \{2\gamma(i) - 1, 2\gamma(i)\}$ for all $i \in \mathbb{Z}_n$.

Example 3.10. See Figure 3, $\eta \in \mathbf{C}_4$ is a doubling of $\gamma \in \mathbf{C}_2$, but $\xi \in \mathbf{C}_4$ is not.

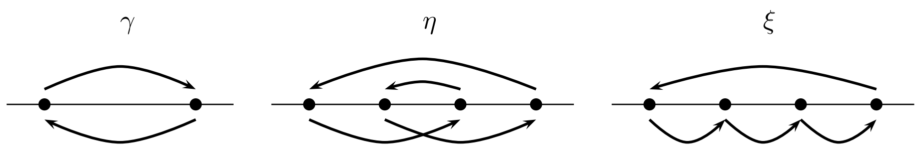

Figure 3

The following result is due to Bernhard [3], and a generalization (Theorem 6.3) will be given in Section 6.

Proposition 3.11. *Let γ, $\theta \in \mathbf{C}$, and η be a doubling of γ. Then $\eta \to \gamma$. Moreover, if $\eta \to \theta$ and $\gamma \neq \theta$, then $\gamma \to \theta$.*

Theorem 3.12. *Let $\Theta \subset \mathbf{C}$, and $\Gamma = \{\gamma_1, \gamma_2, \ldots\} \subset \mathbf{E}(\Theta)$. If for any $n \in \mathbb{N}$ there exists an integer $q(n) > n$ such that $\gamma_n \to \gamma_{q(n)}$, then every $f \in F_{\gamma_1}(I)$ has a reduction $\varphi \in \bigcap\{F_\theta(I) : \theta \in \Theta\} - \bigcup\{F_{\gamma_n}(I) : n \in \mathbb{N}\}$.*

Proof. Note $f \in F_{\gamma_1}(I) \subset F_{\gamma_{q(1)}}(I)$. By Theorem 3.6, f has a periodic orbit Q_1 of pattern $\gamma_{q(1)}$ and a normal reduction f_1 preserving Q_1 such that $f_1 \in F^*_{\gamma_{q(1)}}(L(Q_1))$. Write $\beta_1 = \gamma_{q(1)}$. For $n \geq 2$, assume Q_{n-1}, β_{n-1} and f_{n-1} have been defined satisfying $f_{n-1} \in F^*_{\beta_{n-1}}(L(Q_{n-1}))$. If $f_{n-1} \notin F_{\gamma_n}(L(Q_{n-1}))$, then let $Q_n = Q_{n-1}, \beta_n = \beta_{n-1}$ and $f_n = f_{n-1}$. If $f_{n-1} \in F_{\gamma_n}(L(Q_{n-1}))$, then let $\beta_n = \gamma_{q(n)}$. Since $F_{\gamma_n}(L(Q_{n-1})) \subset F_{\beta_n}(L(Q_{n-1}))$, one can take a periodic orbit Q_n of f_{n-1} with pattern β_n and a normal reduction f_n of f_{n-1} preserving Q_n such that $f_n \in F^*_{\beta_n}(L(Q_n))$. Write $J_n = L(Q_n)$ for $n \geq 1$. By induction, one obtains infinite sequences $\{f_n\}_{n=1}^\infty$, $\{Q_n\}_{n=1}^\infty$, $\{J_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ which satisfy that $J_n = L(Q_n) \supset J_{n+1}, \beta_n \in \{\gamma_{q(i)} : i = 1, 2, \ldots, n\}$ and $f_n \in F^*_{\beta_n}(J_n) - F_{\gamma_n}(J_n)$ is a reduction of f_{n-1} for all $n \geq 1$. Let $J = \bigcap_{n=1}^\infty J_n$. By Proposition 2.4, $f_1|_J, f_2|_J, f_3|_J, \ldots$ converges uniformly to a map $g \in C^0(J)$, which is a reduction of f and of each f_n. By Lemma 2.2, $g \notin \bigcup\{F_{\gamma_n}(J) : n \in \mathbb{N}\}$.

For any $\theta \in \Theta$ and $n \in \mathbb{N}$, since $\beta_n \in \Gamma \subset \mathbf{E}(\Theta) \subset \mathbf{E}(\theta)$, one has $f_n \in F_{\beta_n}(J_n) \subset F_\theta(J_n)$. Let

$$V_n = \{x \in P(f_n) : \text{ the pattern of } \mathcal{O}(x, f_n) \text{ is } \theta\} \text{ and } v_n = \inf V_n.$$

Then $v_n \in P(f_n)$. Since the pattern of Q_n under $f_i (1 \leq i \leq n)$ is β_n and $\beta_n \neq \theta$, one has $V_1 \cap Q_n = \emptyset$. Note that f_n is piecewise monotonic. It follows from

Lemma 3.2 that $v_n \in V_n$. By Lemma 2.2, one has $V_1 \supset V_2 \supset V_3 \supset \dots$. Hence $v_1 \le v_2 \le v_3 \le \dots$. Let $w = \lim_{n\to\infty} v_n$. Then $w \in P(g)$. Suppose the period of θ (and of $\mathcal{O}(v_n, f_n)$) is k. For $i \in \mathbb{N}$, let $v_n^i = f_n^i(v_n)$, and let (c_n^i, d_n^i) be the connected component of $J_n - Q_n$ containing v_n^i. Then $\{c_n^i, d_n^i\} \subset Q_n$, and $f_n|_{[c_n^i, d_n^i]}$ is monotonic.

Suppose the pattern of $\mathcal{O}(w, g)$ is ζ. If $\zeta \ne \theta$, then, by Lemma 3.2, θ must be a doubling of ζ and there exists a closed interval $U_\varepsilon = [w-\varepsilon, w+\varepsilon] \subset (c_1^k, d_1^k)$ such that for any $n \ge 1$ and any $x \in U_\varepsilon \cap P(f_n)$, the pattern of $\mathcal{O}(x, f)$ is θ and $f_n^i(U_\varepsilon) \subset [c_n^i, d_n^i]$ for all $i \in \mathbb{Z}_k$. Take $m \in \mathbb{N}$ such that $v_m \in [w-\varepsilon, w)$ and $v_m < v_{m+1} < w$. Then $v_m \in P_k(f_m) - P(f_{m+1})$ and there is $j \in \mathbb{Z}_k$ such that $f_{m+1}(v_m^j) \ne f_m(v_m^j)$. let $[y_1, y_2]$ be the connected component of $f_{m+1}^{-1}(f_{m+1}(v_m^j)) \cap [c_m^j, d_m^j]$ containing v_m^j. Since f_{m+1} is a reduction of f_m, by Definition 2.1 one has $c_m^j \le y_1 < v_m^j < y_2 \le d_m^j$ and

$$f_m(y_1) = f_{m+1}(y_1) = f_{m+1}(y_2) = f_m(y_2) = f_{m+1}(v_m^j) \ne f_m(v_m^j).$$

This implies that $f_m|_{[c_m^j, d_m^j]}$ is not monotonic, which yields a contradiction. Thus $\zeta = \theta$ and $g \in F_\theta(J)$.

To sum up, we have proved that $g \in \bigcap\{F_\theta(J) : \theta \in \Theta\} - \bigcup\{F_{\gamma_n}(J) : n \in \mathbb{N}\}$. Suppose $J = [c, d]$. Define $\varphi \in C^0(I)$ by

$$\varphi(x) = g(\max\{c, \min\{x, d\}\}), \quad \text{for any} \quad x \in I.$$

Then φ is also a reduction of f and $\varphi \in \bigcap\{F_\theta(I) : \theta \in \Theta\} - \bigcup\{F_{\gamma_n}(I) : n \in \mathbb{N}\}$. □

3.3. **The converse of Sharkovskiĭ's theorem.** To conclude this section, we go back to the converse of Sharkovskiĭ's theorem. Let $F(I, 2^\infty)$ and $\Phi(I, 2^\infty)$ be the same as in Theorem 1.2. By an argument analogous to the proof of Theorem 3.12, one has

Theorem 3.13. *Every $f \in \Phi(I, 2^\infty)$ has a reduction $\varphi \in F(I, 2^\infty) - \Phi(I, 2^\infty)$.*

According to Theorem 3.13 one can obtain a family of maps in $F(I, 2^\infty) - \Phi(I, 2^\infty)$, which are distinct from those given by Delahaye [18].

4. Patterns of invariant sets of interval maps

In this section we introduce the definition of patterns of invariant sets of interval maps, and give some general results on the conditions under which one pattern can force another one. In the sequel, we will use the results of this section to study some special patterns.

4.1. **Patterns of invariant sets of interval maps.** Firstly we introduce some notions. Recall that a pair (X, ψ) is called a **line system** if X is a nonempty bounded subset of $\mathbb{R}$ with the usual metric, ψ is a continuous map from X to X, and ψ can be extended to an interval map. Note that if (X, ψ) is a line system then there exists a unique continuous map $\overline{\psi} : \overline{X} \to \overline{X}$, called the **closure extension** of ψ, such that $\overline{\psi}|_X = \psi$. A line system (X, ψ) is said to be **compact** if X is compact. Denote by Ψ (resp. Ψ_c) the family of all line systems (resp. compact line systems).

For any nonempty subsets X and Y of $\mathbb{R}$, an injection $h : X \to Y$ is said to be **order-preserving** if $h(x) < h(y)$ for all $x, y \in X$ with $x < y$. It is clear that if both X and Y are compact then every order-preserving bijection from X to Y is a homeomorphism.

Definition 4.1. Let (X,ψ) and (Y,ξ) be two line systems. We say that (X,ψ) and (Y,ξ) have the same **pattern** (for convenience, sometimes we also say that ψ and ξ have the same pattern) if there exists an order-preserving bijection $h : X \to Y$ such that $h\psi = \xi h$.

If (X,ψ) and (Y,ξ) have the same pattern, then denote it by $(X,\psi) \approx (Y,\xi)$, or $\psi \approx \xi$ when it is possible without ambiguity.

Example 4.2. Let $X = [-1,1], J = [-2,2]$, and let $\psi : X \to X$ and $f : J \to J$ be strictly decreasing continuous maps satisfying

$$\psi(x) = -x, \qquad \text{if } \ x \in [-1,0] \cup \{1/n : n \in \mathbb{N}\};$$

$$\psi(x) > -x, \qquad \text{if } \ x \in (\frac{1}{n+1}, \frac{1}{n}), \ n \in \mathbb{N}.$$

$$f(y) = \begin{cases} -y, & \text{if } y \in [-2,1]; \\ \psi(y-1)-1, & \text{if } y \in [1,2]. \end{cases}$$

Let $Y = [-2,-1) \cup \{0\} \cup (1,2]$ and $\xi = f|_Y$. Then (X,ψ) and (Y,ξ) have the same pattern. Note that their closure extensions $(\overline{X},\overline{\psi})$ and $(\overline{Y},\overline{\xi})$ do not have the same pattern. Since X is compact but Y is not, there is no homeomorphism $H : X \to Y$ such that $H\psi = \xi H$, and hence (X,ψ) and (Y,ξ) are not topologically conjugate.

Remark 4.3. Because of this example and many other analogous examples, in Definition 4.1 we do not require X and Y to be compact nor do we require the order preserving bijection h to be a homeomorphism.

The reason why we have to focus on Ψ but not only Ψ_c is the phenomenon happened in Example 4.2 (and Example 4.7 etc.) but not just because Ψ is a bigger family than Ψ_c.

Denote by Ψ^* the set of equivalence classes in Ψ under the equivalence relation $\approx$, i.e. $\Psi^* = \Psi/\approx$. Then Ψ^* can be regarded as the set of patterns of invariant sets of interval maps. For $(X,\psi) \in \Psi$, one uses $[(X,\psi)]$ to denote the equivalence class containing (X,ψ). It is easy to see that one can regard $\mathbf{C}$ as a subset of Ψ^*.

4.2. **Forcing relation.** Let $(X,\psi) \in \Psi$, $I = [a,b]$ and $f : I \to I$ be an interval map. f is said to **have an invariant set** S **with the pattern of** (X,ψ) or f **exhibits** (X,ψ) on S if there exists an f-invariant set S such that $f|_S$ and ψ have the same pattern, i.e. $(S, f|_S) \approx (X,\psi)$.

Definition 4.4. Let (X,ψ), $(Y,\xi) \in \Psi$. We say $[(X,\psi)]$ **forces** $[(Y,\xi)]$ if if each interval map exhibiting (X,ψ) also exhibits (Y,ξ). Denote it by $[(X,\psi)] \Rightarrow [(Y,\xi)]$, or $(X,\psi) \Rightarrow (Y,\xi)$ and $\psi \Rightarrow \xi$ when there is no confusion.

So we obtain a transitive relation $\Rightarrow$ on Ψ^*. The relation $\to$ defined in Section 3 is a restriction of $\Rightarrow$ to $\mathbf{C}$. Hence, conversely, the relation $\Rightarrow$ is an extension of $\to$. Note that in Definition 4.4 we do not insist $(X,\psi) \neq (Y,\xi)$, thus we have $(X,\psi) \Rightarrow (X,\psi)$ for all $(X,\psi) \in \Psi$.

Example 4.5. Let $\psi : \mathbb{Z}_3 \to \mathbb{Z}_3$ and $\xi_n : \mathbb{Z}_n \to \mathbb{Z}_n$ be defined by $\psi(1) = \psi(3) = 1$, $\psi(2) = 3$, $\xi_n(n) = 1$, and $\xi_n(i) = i+1$ for $1 \le i < n$. Then it is well known that ψ forces ξ_n for all $n \ge 1$.

Example 4.6. Let $X = \{0, 1/2, 1\}$ and $\psi : X \to X$ be defined by $\psi(0) = 1/2, \psi(1/2) = 1$ and $\psi(1) = 0$. Let $Y = [0,1]$ and $\xi : Y \to Y$ satisfy that $\xi|_X = \psi$ and ξ is linear on interval $[0,1/2], [1/2,1]$ (See Figure 4). Then $[(X,\psi)] \neq [(Y,\xi)]$, but $[(X,\psi)] \Rightarrow [(Y,\xi)]$ and $[(Y,\xi)] \Rightarrow [(X,\psi)]$. Hence the forcing relation on Ψ^* is not a partial ordering.

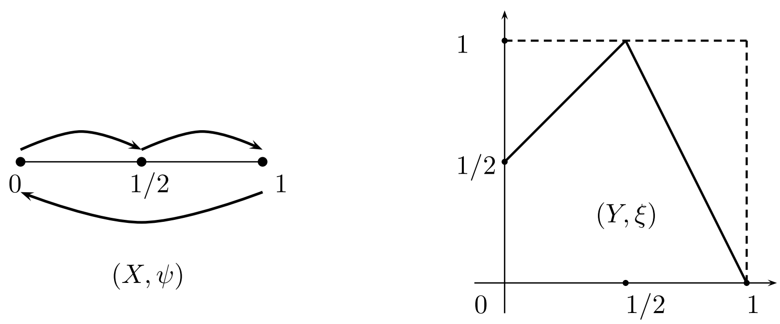

Figure 4

Example 4.7. Let $X = \{-3,-2,0,2,3\}$, $Y = \{-3,0,3\} \cup \{b_n, -b_n : n \in \mathbb{N}\}$ and $W = \{-3,0,3\} \cup \{c_n, -c_n : n \in \mathbb{N}\}$, where

$$b_n = 2^{-n}, \qquad c_n = 1 + 2^{-n}, \qquad \text{for all} \quad n \in \mathbb{N}.$$

Define $\psi \in C^0(X)$, $\xi \in C^0(Y)$ and $\eta \in C^0(W)$ by

$$\begin{aligned}
&\psi(0) = \xi(0) = \eta(0),\\
&\psi(2) = \psi(3) = \xi(b_1) = \xi(3) = \eta(c_1) = \eta(3) = 3,\\
&\psi(-2) = \psi(-3) = \xi(-b_1) = \xi(-3) = \eta(-c_1) = \eta(-3) = -3 \qquad \text{and}\\
&\xi(b_{n+1}) = b_n, \quad \xi(-b_{n+1}) = -b_n, \quad \eta(c_{n+1}) = c_n, \quad \eta(-c_{n+1}) = -c_n
\end{aligned}$$

for all $n \in \mathbb{N}$. Let $Y' = Y - \{0\}$, $W' = W - \{0\}$, and let $\xi' = \xi|_{Y'}$, $\eta' = \eta|_{W'}$. Then $\{(X,\psi), (Y,\xi), (W,\eta), (Y',\xi'), (W',\eta')\} \subset \Psi$. It is easy to see that

$$(X,\psi) \Rightarrow (Y,\xi) \approx (W,\eta) \Rightarrow (Y',\xi') \approx (W',\eta') \Rightarrow (X,\psi). \tag{4.1}$$

Since $[(X,\psi)]$, $[(Y,\xi)]$ and $[Y',\xi')]$ are pairwise unequal, from (4.1) we see that the relation $\Rightarrow$ is not a partial order on Ψ^*. Note that X and Y are compact, and W, Y' and W' are not compact. Obviously, there exists an interval map f such that (X,ψ) is a subsystem of f and hence f exhibits (Y,ξ), but f has no compact invariant set with the pattern of (Y,ξ). This example also explains why we have to consider the relation $\approx$ in Ψ but not only in Ψ_c.

4.3. **A useful criterion for the forcing relation.** In the rest of this section we will give some conditions of a pattern forcing another. Theorem 4.14 is the main result of this section, and Theorem 4.12 is a useful criterion. We begin with some notions and notations used in the sequel.

Definition 4.8. Let $(X,\psi)\in\Psi$ and $I=L[X]=[\inf X,\sup X]$. A map $g\in C^0(I)$ is called a **monotonic extension** (resp. the **linear extension**) of ψ if $g|_X=\psi$ and g is monotonic (resp. linear) on every connected component of $I-X$. A map $g\in C^0(I)$ is called a **strictly monotonic extension** of ψ if $g|_X=\psi$, and for any connected component $J=(v,y)$ of $I-\overline{X}$, when $g(v)\neq g(y)$ then $g|_J$ is strictly monotonic, and when $g(v)=g(y)$ then $g|_J$ is constant.

Note that any map $g\in C^0(L(S))$ is both monotonic and linear on every singleton in $L(S)$. Thus, g is a monotonic (resp. strictly monotonic, resp. linear) extension of ψ if and only if g is a monotonic (resp. strictly monotonic, resp. linear) extension of the closure extension $\overline{\psi}$. Note that, for each line system (X,ψ), ψ has a unique linear extension.

Remark 4.9. Let X and Y be nonempty bounded subsets of $\mathbb{R}$ with an order-preserving bijection $h:X\to Y$, let K be a connected component of $L[X]-X$ and $L[X]=[a,b]$. It is easy to see that

(1) If $K=(r,s)$ is an open interval, then the corresponding open interval $(h(r),h(s))$ is also connected component of $L[Y]-Y$.

(2) If $K=[r,s]$ is a closed interval, then it is possible that $\sup(h(X\cap[a,r]))=\inf(h(X\cap[s,b]))$. In this case, the connected component of $L[Y]-Y$ corresponding to K is a one point set $\{\sup(h(X\cap[a,r]))\}$.

Conversely, if $K=\{r\}$ is a one point set and $r\notin\{a,b\}$, then it is possible that $\sup(h(X\cap[a,r]))<\inf(h(X\cap[r,b]))$. In this case, the connected component of $L[Y]-Y$ corresponding to K is a closed interval $[\sup(h(X\cap[a,r])),\inf(h(X\cap[r,b]))]$.

(3) If K is a semi-open interval, say, $K=(r,s]$, then it is possible that $h(r)=\inf(h(X\cap[s,b]))$. In this case, there is no connected component of $L[Y]-Y$ corresponding to K.

(4) If X is compact, then every connected component of $L[X]-X$ is an open interval.

According to Remark 4.9, in the following we only consider the connected components of $L[X]-X$ which are open intervals.

Definition 4.10. Let $X\subset\mathbb{R}$ be a nonempty bounded set, and $I=L(X)$. A connected component K of $I-X$ is called an **open complementary interval** of X if K is an open interval.

Denote by $\mathbf{K}(X)$ the set of all open complementary intervals of X. Write

$$U(X)=\bigcup\{J:J\in\mathbf{K}(X)\}.$$

Then $\mathbf{K}(X)\subset\mathbf{K}(\overline{X})$, $U(X)\subset I-X$, and $U(X)=I-X$ if and only if X is compact. For any $y\in X\cup U(X)$, let

$$K(y,X)=\begin{cases} J, & \text{if } y\in J \text{ and } J\in\mathbf{K}(X); \\ y, & \text{if } y\in X. \end{cases}$$

Let (X,ψ) be a line system. For any monotonic extension f of ψ and any orbit $\mathcal{O}(x,f)$ contained in $X\cup U(X)$, write

$$\mathbf{I}(x,f,X)=(K(x,X),\ K(f(x),X),\ K(f^2(x),X),\ \dots),$$

$$\mathbf{I}_m(x,f,X) = (K(x,X)\,,\ K(f(x),X)\,,\ \ldots,\ K(f^m(x),X))\,,\quad (m \geq 0).$$

The infinite sequence $\mathbf{I}(x,f,X)$ is called the **itinerary of x under f relative to X**. It is easy to see that, if $\mathbf{I}_m(x,f,X) = \mathbf{I}_m(y,f,X)$ but $f^{m+1}(x) \neq f^{m+1}(y)$, then $K(f^i(y)\,,X) = K(f^i(x)\,,X) \subset U(X)$ for $0 \leq i \leq m$.

For any nonempty subsets X and Y of $\mathbb{R}$, one writes $X < Y$ if $x < y$ for any $x \in X$ and any $y \in Y$.

Lemma 4.11. *Let $X, U(X)$ and f be as above. Suppose $\mathcal{O}(x,f)$ and $\mathcal{O}(y,f)$ are two orbits contained in $X \cup U(X)$. Then $x < y$ if one of the following three conditions holds:*

(i) $K(x,X) < K(y,X)$.

(ii) *There is an $m \geq 0$ such that $\mathbf{I}_m(x,f,X) = \mathbf{I}_m(y,f,X), K(f^{m+1}(x),X) < K(f^{m+1}(y),X)$ and the number $|\{0 \leq i \leq m :\ f|_{K(f^i(x),X)} \text{is decreasing}\}|$ is even.*

(iii) *There is an $m \geq 0$ such that $\mathbf{I}_m(x,f,X) = \mathbf{I}_m(y,f,X)$, $K(f^{m+1}(x),X) > K(f^{m+1}(y),X)$, and the number $|\{0 \leq i \leq m :\ f|_{K(f^i(x),X)} \text{ is decreasing}\}|$ is odd.*

The proof of Lemma 4.11 is analogous to that of [17, Lemma II.1.2], and is omitted.

Let $(J_0, J_1, J_2, \ldots)$ be an infinite sequence of open complementary intervals of X such that $J_n = (r_n, s_n)$ and $[\psi(r_n); \psi(s_n)] \supset J_{n+1}$ for all $n \geq 0$. J_0 is said to be **expanding under ψ relative to** $(J_0, J_1, J_2, \ldots)$ if there exist j and $k \in \mathbb{N}$ such that $\psi^j(r_0) \notin \partial J_j$ and $\psi^k(s_0) \notin \partial J_k$.

Recall that an orbit $\mathcal{O}(x,g)$ of a map $g \in C^0(X)$ is said to be **eventually periodic** if it is a finite set. $\mathcal{O}(x,g)$ is called an **infinite orbit** if it is an infinite set. For $m \in \mathbb{N}$, an infinite sequence $(K_0, K_1, K_2, \ldots,)$ is said to be periodic and have period m if

$$(K_m, K_{m+1}, K_{m+2}, \ldots) = (K_0, K_1, K_2, \ldots) \neq (K_i, K_{i+1}, K_{i+2}, \ldots), \text{for } 1 \leq i < m.$$

The following theorem is a useful criterion for the forcing relation.

Theorem 4.12. *Let $(X,\psi), (Z,\xi)$ be line systems. Suppose that there exists a monotonic extension f of ψ satisfying the following conditions:*

(C.1) *f has an invariant set $V \subset X \cup U(X)$ such that $f|_V$ and ξ have the same pattern;*

(C.2) *$\mathbf{I}(x,f,X) \neq \mathbf{I}(y,f,X)$ for any $x,y \in V \cap U(X)$ with $x \neq y$;*

(C.3) *For any $x \in V \cap U(X)$, if $\mathcal{O}(x,f)$ is an infinite orbit contained in $U(X)$ or is a periodic orbit in an even state, then the open complementary interval $K(x,X)$ is expanding under ψ relative to $(K(x,X), K(f(x),X), K(f^2(x),X), \ldots)$.*

(C.4) *For any $J = (v,y) \in \mathbf{K}(X)$, if $\psi(v) = \psi(y)$ then $V \cap J = \emptyset$.*

Then $[(X,\psi)] \Rightarrow [(Z,\xi)]$.

Proof. Consider any $g \in C^0(I)$, where I is a compact interval. Assume g has an invariant set Y with the pattern of ψ. Then there is an order preserving bijection $h : X \to Y$ such that $gh = h\psi$. In order to prove $\psi \Rightarrow \xi$, it suffices to show that g has an invariant set V' with the pattern of $f|_V$. Suppose $\mathbf{K}(X) = \{J_i = (r_i, s_i) : i \in Z'\}$, where Z' is a subset of $\mathbb{N}$. For every $i \in Z'$, let $r'_i = h(r_i),\ s'_i = h(s_i)$, and $J'_i = (r'_i, s'_i)$. Then J'_i is an open complementary interval of Y, $\mathbf{K}(Y) = \{J'_i : i \in Z'\}$ and $U(Y) = \bigcup\{J'_i : i \in Z'\}$. Since $g(Y) \subset Y$, we have $g(\overline{Y}) \subset \overline{Y}$. By Theorem 2.8, there exists a normal reduction φ of g preserving $\overline{Y}$. Obviously, one has

$$\varphi(J'_i) \supset J'_k \text{ if and only if } f(J_i) \supset J_k, \quad i, k \in Z'. \tag{4.2}$$

Let $U' = U(Y)$, and $U'' = \{x \in U' : \varphi(x) = g(x)\}$. Then, by Definition 2.1 and Lemma 2.2, φ is constant on every connected component of $U' - U''$, $P(\varphi) \cap U' \subset U''$ and

$$\varphi(\partial J'_i \cup (J'_i \cap U'')) = \varphi(\overline{J'_i}), \quad \text{for any } i \in Z'. \tag{4.3}$$

We are going to show that there exists an order preserving injection $H : V \to Y \cup U''$ such that $Hf|_V = \varphi H (= gH)$. Define $\beta : X \cup \mathbf{K}(X) \to Y \cup \mathbf{K}(Y)$ by $\beta(x) = h(x)$ for any $x \in X$ and $\beta(J_i) = J'_i$ for any $i \in Z'$. Then β is an order preserving bijection. For any sequence $(K_0, K_1, \ldots,)$ of elements in $X \cup \mathbf{K}(X)$ and any $n \in \mathbb{N}$, let

$$\beta((K_0, K_1, \ldots, K_n)) = (\beta(K_0), \beta(K_1), \ldots, \beta(K_n)),$$

$$\beta((K_0, K_1, K_2, \ldots)) = (\beta(K_0), \beta(K_1), \beta(K_2), \ldots).$$

Let $V_0 = V \cap X$. For any $v \in V_0$, define $H(v) = h(v)$. Then

$$\mathbf{I}(H(v), \varphi, Y) = \beta(\mathbf{I}(v, f, X)). \tag{4.4}$$

Let $V_1 = P(f|_V) \cap U(X)$. Take a subset $V_2 \subset V_1$ such that, for every periodic orbit Q of f contained in $U(X)$, $V_2 \cap Q$ contains exactly one point. For any $v \in V_2$, we claim that there is a point $v' \in U'' \cap P(\varphi)$ (and then we define $H(v) = v'$) such that (4.4) holds. In fact, suppose the period of v under f is m, and $\mathbf{I}(v, f, X) = (J_{i(0)}, J_{i(1)}, J_{i(2)}, \ldots)$. Then, by the condition (C. 2), the period of $(J_{i(0)}, J_{i(1)}, J_{i(2)}, \ldots)$ is also m. If $\mathcal{O}(v, f)$ is in an odd state, then it follows from (4.2) that there exists a point $y \in J'_{i(0)}$ such that $\mathbf{I}_{2m}(y, \varphi, Y) = (J'_{i(0)}, J'_{i(1)}, J'_{i(2)}, \ldots, J'_{i(2m)})$ and $(\varphi^m(y) - y) \cdot (\varphi^{2m}(y) - \varphi^m(y)) \leq 0$. Hence there is a point $v' = H(v) \in [y; \varphi^m(y)] \cap P_m(\varphi)$ satisfying (4.4). If $\mathcal{O}(v, f)$ is in an even state, then it follows from the condition (C. 3) that there exist q and $t \in J'_{i(0)}$ with $q < t$ such that $\varphi^m(q) = r'_{i(0)}$, $\varphi^m(t) = s'_{i(0)}$, and $\varphi^n((q, t)) \subset J'_{i(n)}$ for all $n \in \mathbb{Z}_m$. Thus there is also a point $v' = H(v) \in (q, t) \cap P_m(\varphi)$ satisfying (4.4).

For any $v \in V_1 - V_2$, take $u \in V_2$ and $k \geq 1$ such that $v = f^k(u)$, and put $H(v) = \varphi^k(H(u))$. Obviously, for such v and $H(v)$, (4.4) still holds.

Let $V_3 = \{x \in V : \mathcal{O}(x, f) \subset U(X) - P(f)\}$. Take $V_4 \subset V_3$ such that $\bigcup_{p=0}^\infty \bigcup_{n=0}^\infty f^{-p}(f^n(V_4)) \supset V_3$ and $\mathcal{O}(x, f) \cap \mathcal{O}(y, f) = \emptyset$ for any $x, y \in V_4$ with $x \neq y$. For any $v \in V_4$ and any $n \in \mathbb{N}$, it follows from (4.2) and (4.3) that there exists $v'_n \in U''$ such that $\mathcal{O}_n(v'_n, \varphi) \subset U''$ and $\mathbf{I}_n(v'_n, \varphi, Y) = \beta(\mathbf{I}_n(v, f, X))$. Let $v' = H(v) = \liminf_{n\to\infty} v'_n$. Then by condition (C.3) it is easy to check that $\mathcal{O}(v', \varphi) \subset U''$, and hence (4.4) holds for $v \in V_4$.

Let $V_5 = \bigcup_{n=0}^\infty f^n(V_4)$. For any $u \in V_4$, $n \in \mathbb{N}$ and $v = f^n(u) \in V_5$, let $v' = H(v) = \varphi^n(H(u))$. Then $\mathcal{O}(v', \varphi) \subset \mathcal{O}(H(u), \varphi) \subset U''$, and (4.4) holds for $v \in V_5$.

Finally, for $n = 1, 2, 3, \ldots$ and for every $w \in f^{-n}(V_0 \cup V_1 \cup V_5) \cap V - f^{-n+1}(V_0 \cup V_1 \cup V_5)$, if $H(f(w))$ has been defined, then by (4.2) and (4.3) and condition (C.4) one can continue to choose a point $w' = H(w) \in U''$ such that

$$\varphi(H(w)) = H(f(w)) \quad \text{and} \quad \mathbf{I}(H(w), \varphi, Y) = \beta(\mathbf{I}(w, f, X)).$$

Therefore, noting $V = \bigcup_{n=0}^{\infty}(f^{-n}(V_1 \cup V_2 \cup V_5) \cap V)$, one obtains a map $H : V \to U'' \cup Y$ which satisfies $Hf|_V = \varphi H$, $H(P_m(f|_V)) \subset P_m(\varphi)$ for any $m \in \mathbb{N}$, and

$$\mathbf{I}(H(x), \varphi, Y) = \beta(\mathbf{I}(x, f, X)) \text{ for any } x \in V. \tag{4.5}$$

By (4.5), condition (C.2) and Lemma 4.11, H is an order preserving injection. Let $V' = H(V)$. Then $g|_{V'}(= \varphi|_{V'})$ and $f|_V$ have the same pattern. So the proof of Theorem 4.12 is completed. □

4.4. **Main results of this section.**

Definition 4.13. Let (V, ξ) be a line system, and $x, y \in V$ with $x \neq y$. The orbits $\mathcal{O}(x, \xi)$ and $\mathcal{O}(y, \xi)$ are said to be **companionate orbits of** ξ if

(i) $\xi|_{[\xi^k(x);\xi^k(y)]\cap V}$ is monotonic for all $k \geq 0$;

(ii) for $j > k \geq 0$, if $[\xi^j(x); \xi^j(y)] \cap [\xi^k(x); \xi^k(y)]$ contains more than one points, then $[\xi^j(x); \xi^j(y)] = [\xi^k(x); \xi^k(y)]$.

Two companionate orbits $\mathcal{O}(x, \xi)$ and $\mathcal{O}(y, \xi)$ are said to be **self-companionate** if $\mathcal{O}(y, \xi) \subset \mathcal{O}(x, \xi)$ or $\mathcal{O}(x, \xi) \subset \mathcal{O}(y, \xi)$.

It is easy to check that if $\mathcal{O}(x, \xi)$ and $\mathcal{O}(y, \xi)$ are companionate infinite orbits then they have the same pattern. And if $\mathcal{O}(x, \xi)$ and $\mathcal{O}(y, \xi)$ are self-companionate periodic orbits of period n, then n is even, $y = \xi^{n/2}(x)$ and $[\xi^i(x); \xi^i(y)] \cap [\xi^j(x); \xi^j(y)] = \emptyset$ for $0 \leq i < j < n/2$.

Theorem 4.14. *Let (X, ψ) and (Z, ξ) be line systems, and X be compact. Suppose ξ has no companionate orbits. Then $[(X, \psi)] \Rightarrow [(Z, \xi)]$ if and only if there exists a monotonic extension of (X, ψ) which exhibits ξ.*

Proof. The necessity is clear. We now prove the sufficiency. Let $U(X)$ and $I = [\inf X, \sup X]$ be the same as in Definition 4.8. Since X is compact, we have $I = X \cup U(X)$. Assume there is a monotonic extension f of ψ which has an invariant set W with the pattern of ξ. In order to prove $\psi \Rightarrow \xi$, it suffices to show that there exists an order-preserving injection $H : W \to I$ such that $fH = Hf|_W$ and the conditions (C.2)−(C.4) in Theorem 4.12 hold for $V = H(W)$. Let

$W_0 = \{v \in W : \mathcal{O}(v, f) \subset U(X)\}$,

$V_1 = \{v \in W_0 :$ for any $n \geq 0, K(f^n(v), X)$ is expanding under ψ relative to $(K(f^n(v), X), K(f^{n+1}(v), X), K(f^{n+2}(v), X), \ldots)\}$,

$V_2 = \{v \in W_0 :$ there exists a $j \geq 0$ such that $\mathcal{O}(f^j(v), f)$ is a periodic orbit in an odd state$\}$,

$V_3 = \{v \in W_0 : \mathcal{O}(v, f)$ is an infinite orbit and $K(v, X)$ is not expanding under ψ relative to $(K(v, X), K(f(v), X), K(f^2(v), X), \ldots)\}$,

$V_4 = \{v \in W_0 : \mathcal{O}(v, f)$ is a periodic orbit in an even state and $K(v, X)$ is not expanding under ψ relative to $(K(v, X), K(f(v), X), K(f^2(v), X), \ldots)\}$,

$V_5 = \bigcup_{i=0}^{\infty} f^{-i}(V_3) \cap W$,

$V_6 = \bigcup_{i=0}^{\infty} f^{-i}(V_4) \cap W$,

and

$W_1 = \{v \in W : \mathcal{O}(v, f) \cap X \neq \emptyset\}$,

$V_7 = W_1 \cap X \ (= W \cap X)$,

$V_8 = f^{-1}(V_7) \cap W_1 - V_7 \ (= f^{-1}(V_7) \cap W \cap U(X))$,

$V_{81} = \{v \in V_8 : f|_{K(v,X)}$ is not constant $\}$,

$V_{82} = \{v \in V_8 : f|_{K(v,X)}$ is constant $\}$,
$V_{91} = \bigcup_{i=0}^{\infty} f^{-i}(V_{81}) \cap W_1$,
$V_{92} = \bigcup_{i=0}^{\infty} f^{-i}(V_{82}) \cap W_1$.

Then one has $W_0 \cup W_1 = W$, $W_0 \cap W_1 = \emptyset$, $V_3 \subset V_5$, $V_4 \subset V_6$, $\bigcup_{i=1}^{6} V_i = W_0$, $V_{81} \subset V_{91}$, $V_{82} \subset V_{92}$, and $V_7 \cup V_{91} \cup V_{92} = W_1$. Note that $V_1 \cup V_2, V_5, V_6, V_7, V_{91}$ and V_{92} are pairwise disjoint, and $V_1 \cup V_2, V_3, V_4, V_5, V_6, V_7, V_7 \cup V_{91}$ and $V_7 \cup V_{92}$ are all invariant under f.

Take $V_{30} \subset V_3$ and $V_{40} \subset V_4$ such that $\bigcup_{i=0}^{\infty} \bigcup_{n=0}^{\infty} f^{-i}(f^n(V_{j0})) \supset V_{j+2}$, $(j = 3, 4)$, and $\mathcal{O}(x, f) \cap \mathcal{O}(y, f) = \emptyset$ for any $\{x, y\} \subset V_{30} \cup V_{40}$ with $x \neq y$. Denote $V_{31} = \bigcup\{\mathcal{O}(v, f) : v \in V_{30}\}$. Then $\bigcup_{i=0}^{\infty} f^{-i}(V_{31} \cup V_4) \supset V_5 \cup V_6$.

We now define a map $H : W \to I$ as follows:

Step 1. For any $v \in V_1 \cup V_2 \cup V_7 \cup V_{91}$, let $H(v) = v$.

Step 2. For any $v \in V_{30} \cup V_{40}$, since $K(v, X)$ is not expanding under ψ relative to $(K(v, X), K(f(v), X), \ldots)$, there exists $y_v \in \partial K(v, X)$ such that $\psi^n(y_v) \in \partial K(f^n(v), X)$ for all $n \geq 0$. Note that if $v \in V_{40}$ and if the period of v is k, then $\psi^k(y_v) = f^k(y_v) = y_v$ since $\mathcal{O}(v, f)$ is in an even state. Thus we can put $H(v) = y_v$, and put $H(f^n(v)) = \psi^n(y_v)$ $(= f^n(H(v))$ for all $n \geq 1$. Hence $H|_{(V_{31} \cup V_4)}$ is defined.

Step 3. For $n = 1, 2, 3, \ldots$ and for every $v \in f^{-n}(V_{31} \cup V_4 \cup V_7) \cap W - f^{-n+1}(V_{31} \cup V_4 \cup V_7)$, if $H(f(v))$ has been defined and $H(f(v)) \in \overline{K(f(v), X)}$, then we can take a point $v' \in \overline{K(v, X)}$ such that $f(v') = H(f(v))$, (particularly, if $n = 1$ and $v \in V_{82}$, we take $v' \in \partial K(v, z)$; if $v \in V_{91}$, we take $v' = v$) and then we put $H(v) = v'$.

From these three steps we obtain a map $H : W \to I$ which satisfies

$$fH = Hf|_W \quad \text{and} \quad H(v) \in \overline{K(v, X)} \quad \text{for all} \quad v \in W. \tag{4.6}$$

Now we prove H is an order-preserving injection. We divide the proof into several claims.

Claim 1. $f|_W$ has no companionate orbits.

Since $f|_W$ and ξ have the same pattern, and ξ has no companionate, we have Claim 1.

Claim 2. If $v \in V_{31}$, then $f^n([H(v); v]) = [Hf^n(v); f^n(v)]$ and $[Hf^n(v); f^n(v)] \cap [H(v); v] = \emptyset$ for each $n \in \mathbb{N}$.

Proof of Claim 2. Note that $\mathcal{O}(v, f)$ is an infinite orbit in $U(X)$, f is monotonic on every connected component of $U(X)$, and $Hf^i(v) = f^i H(v) \in X$ for all $i \geq 0$. Thus $f^n([H(v); v]) = [Hf^n(v); f^n(v)]$ for $n \in \mathbb{N}$. Write $v_i = f^i(v)$. If Claim 2 is not true, then there is a minimal positive integer n such that $[H(v_n); v_n] \cap [H(v); v] \neq \emptyset$. This implies that one of the following three cases holds:

Case 1. $H(v_n) = H(v)$ and $(H(v_n); v_n] \cap (H(v); v] \neq \emptyset$. In this case, for $i = 1, 2, 3, \ldots$, one has $H(v_{in}) = H(v)$, and

$$\begin{aligned} & (H(v_{in}); v_{in}] \cap (H(v_{in-n}); v_{in-n}] \\ = \quad & f^{(i-1)n}([H(v_n); v_n]) \cap f^{(i-1)n}([H(v); v]) - \{H(v)\} \neq \emptyset \end{aligned}$$

It follows that $(H(v_{in}); v_{in}] \cap (H(v); v] \neq \emptyset$ and $(v, v_n, v_{2n}, v_{3n}, \ldots)$ is a strictly monotonic sequence of points in $\bigcup_{i=0}^{\infty}(H(v); v_{in}] \subset K(v, X)$. Obviously, for $0 \leq k < j < n$, $(v_j, v_{n+j}, v_{2n+j}, \ldots)$ is also a strictly monotonic sequence in $\bigcup_{i=0}^{\infty}(H(v_j); v_{in+j}]$

$\subset K(v_j, X)$ and $(\bigcup_{i=0}^{\infty}(H(v_k); v_{in+k}]) \bigcap (\bigcup_{i=0}^{\infty}(H(v_j); v_{in+j}]) = \emptyset$. Hence $\mathcal{O}(v, f)$ and $\mathcal{O}(v_n, f)$ are companionate orbits. But this contradicts Claim 1.

Case 2. $H(v_n) \neq H(v)$ and $(H(v_n); v_n] \cap (H(v); v] \neq \emptyset$. In this case, letting w be another endpoint of the open interval $K(v, X)$ except $H(v)$, we have $H(v_n) = w$, $v_n \in (H(v); v)$, $v \in (v_n; w)$ and $H(v_{2n}) = H(v)$. Similar to Case 1, it is easy to see that $(v_j, v_{2n+j}, v_{4n+j}, v_{6n+j}, \ldots)$ is a strictly monotonic sequence in $\bigcup_{i=0}^{\infty}(H(v_{n+j}); v_{2in+j}] \subset K(v_{n+j}, X) = K(v_j, X)$ for $0 \leq j < \infty$ and

$$(\bigcup_{i=0}^{\infty}(H(v_{n+k}); v_{2in+k}]) \cap (\bigcup_{i=0}^{\infty}(H(v_{n+j}); v_{2in+j}]) = \emptyset \quad \text{for} \quad 0 \leq k < j < 2n.$$

Hence $\mathcal{O}(v, f)$ and $\mathcal{O}(v_{2n}, f)$ are companionate orbits. But this also contradicts Claim 1.

Case 3. $H(v_n) = H(v)$ and $(H(v_n); v_n] \cap (H(v); v] = \emptyset$. In this case we have $H(v_{2n}) = H(v)$. If $(H(v_{2n}); v_{2n}] \cap (H(v); v] \neq \emptyset$, then similar to Case 2. it is easy to check that $\mathcal{O}(v, f)$ and $\mathcal{O}(v_{2n}, f)$ are companionate orbits. If $(H(v_{2n}); v_{2n}] \cap (H(v); v] = \emptyset$, then $(H(v_{2n}); v_{2n}] \cap (H(v_n); v_n] \neq \emptyset$, and similar to Case 1. it is easy to check $\mathcal{O}(v_n, f)$ and $\mathcal{O}(v_{2n}, f)$ are companionate orbits. But both still contradict Claim 1. Therefore, we have Claim 2.

Claim 3. Let $v \in V_4$ with period m. Then $f^i([H(v); v]) = [Hf^i(v); f^i(v)]$ for $i = 1, 2, 3, \ldots$, $f^m([H(v); v]) = [H(v); v]$ and

$$[Hf^n(v); f^n(v)] \cap [H(v); v] = \emptyset \quad \text{for each} \quad n \in \{1, \ldots, m-1\}.$$

The proof of Claim 3 is analogous to that of Claim 2, and is omitted.

Claim 4. For any $v, y \in W$,

$$(v; H(v)] \cap W = \emptyset, \tag{4.7}$$

and

$$H(y) \neq H(v), \quad \text{if} \quad y \neq v. \tag{4.8}$$

Proof of Claim 4. Let $Y_0 = V_1 \cup V_2 \cup V_7 \cup V_{91}$, $Y_1 = Y_0 \cup V_{31} \cup V_4 \cup V_{82}$ and $Y_i = f^{-1}(Y_{i-1}) \cap W$ $(i = 2, 3, 4, \ldots)$. Then

$$f(Y_{j+1}) \subset Y_j \subset Y_{j+1}, \quad \text{for} \quad j = 0, 1, 2, \ldots, \tag{4.9}$$

and

$$\bigcup_{j=0}^{\infty} Y_j = W. \tag{4.10}$$

If $\{v, y\} \subset Y_0$, then from Step 1 of the definition of H one has that (4.7) and (4.8) hold.

If $v \in V_{31} \cup V_4 \cup V_{82}$ and $y \in Y_1$, then, by Claim 1$-$3, it is easy to check that (4.7) and (4.8) hold.

We now assume that (4.7) and (4.8) hold for all $\{v, y\} \subset Y_n$, where $n \geq 1$. If (4.7) does not hold for some $v \in Y_{n+1}$, then there is a $w \in (v; H(v)] \cap W$. Since $f|_{[v, H(v)]}$ is monotonic, $f(w) \in [f(v); Hf(v)] \cap W$. This with $(f(v); Hf(v)] \cap W = \emptyset$ implies $f(w) = f(v)$, which leads to $f([w; v]) = \{f(v)\}$. If there exists a minimal integer $k \geq 0$ such that $f^k(f(v)) \in (w; v)$, we put $z = f^{k+1}(v)$. If $\mathcal{O}(f(v), f) \cap (w; v) = \emptyset$, we put $z = w$. Then $\mathcal{O}(z, f)$ and $\mathcal{O}(v, f)$ will be companionate orbits contained in W. But this contradicts Claim 1. Thus (4.7) still holds for all $v \in Y_{n+1}$.

If (4.8) does not hold for some $v, y \in Y_{n+1}$, i.e. there exist y and v in Y_{n+1} with $y \neq v$ such that $H(y) = H(v)$, then $Hf(y) = fH(y) = fH(v) = Hf(v)$. Since $\{f(y), f(v)\} \subset Y_n$, by the assumption, $f(y) = f(v)$. Noting that it has been proved that $(v; H(v)] \cap W = \emptyset$ and $(y; H(y)] \cap W = \emptyset$, one has $(v; y) \cap W = \emptyset$. Thus $\mathcal{O}(v, f)$ and $\mathcal{O}(y, f)$ are companionate orbits. But this contradicts Claim 1. Hence (4.8) also holds for all $v, y \in Y_{n+1}$.

By induction, (4.7) and (4.8) hold for all $v, y \in W$. Claim 4 is proven.

Let $V = H(W)$. As a direct corollary of Claim 4, one has

Claim 5. $H : W \to I$ is an order-preserving injection. Thus $f|_V$ and $f|_W$ have the same pattern as ξ.

By Claim 1, $f|_V$ has no companionate orbits, and hence the condition (C.2) in Theorem 4.12 holds. From the step 2 (resp. step 3, the case that $n = 1$ and $v \in V_{82}$) of the definition of H it is easy to see that the condition (C.3) (resp. (C.4)) in Theorem 4.12 holds. Hence, by Theorem 4.12, we have $\psi \Rightarrow \xi$. The proof of theorem is completed. □

Theorem 4.15. *Let $X = \{x_1 < x_2 < \cdots < x_n\}$ be a finite subset of $\mathbf{R}$, $n \geq 3$, $\psi : X \to X$ be a cyclic permutation, and ξ be the linear extension of ψ. If n is odd, or n is even but $O(x_1, \psi)$ and $O(\psi^{n/2}(x_1), \psi)$ are not companionate orbits, then $\psi \Rightarrow \xi$.*

Proof. Take $f = \xi$, and $V = [X]$ $(= [x_1, x_n])$. Then the conditions (C.1) and (C.4) in Theorem 4.12 hold. From the conditions of Theorem 5.15 it is easy to check that there exists a constant number $c > 1$ such that for any $i \in \{2, 3, \cdots, n\}$, $k \geq 1$, and any $\{y, w\} \subset [x_{i-1}, x_i]$, if $\{f^k(y), f^k(w)\} \subset [x_{i-1}, x_i]$ then $|f^k(y) - f^k(w)| \geq c \cdot |y - w|$. This implies that the conditions (C.2) and (C.3) also hold. Hence, by Theorem 4.12 we obtain Theorem 4.15. □

5. Periodic and non-periodic minimal patterns

In Section 4 we give some general results on conditions under which one pattern can force another. It is nature to ask whether one can weaken the condition when a pattern has some special form? In this section we study the periodic and non-periodic minimal patterns.

5.1. **Periodic patterns.** It is well known that for any η and $\theta \in \mathbf{C}$ with $\eta \neq \theta$, η forces θ if and only if the linear extension of η has a periodic orbit of pattern θ (see [1, 2, 10, 23]). The following Theorem 5.2 is a generalization of this result. To prove Theorem 5.2, one need the following lemma.

Lemma 5.1. *Let $I = [a, b]$, $f \in C^0(I)$, $\theta \in \mathbf{C}_n$ $(n \geq 1)$ and $S \supset \{a, b\}$ be a nonempty closed invariant set of f. Suppose f is monotonic on every connected component of $I - S$ and $f|_S$ has no periodic orbit of pattern θ. If f has a periodic orbit $W = \{w_1 < w_2 < \ldots < w_n\}$ of pattern θ in an even state, then f has a periodic orbit $V = \{v_1 < v_2 < \ldots < v_n\}$ of pattern θ in an odd state such that $v_n > w_n$ and*

$$\mathcal{O}(x, f) \not\subset [a, w_n), \quad \text{for any } x \in \cup_{i=1}^{n} [w_i;\ v_i], \tag{5.1}$$

$$[w_i; v_i] \cap [w_j; v_j] = \emptyset, \quad \text{for } 1 \leq i < j \leq n. \tag{5.2}$$

Proof. Let $w = w_n$ and

$$X_0 = \{x \in [w,b] : f^n(x) \geq w\}. \tag{5.3}$$

Then X_0 is closed. Take $\varepsilon > 0$ such that $[w-3\varepsilon, w+3\varepsilon] \subset [w_{n-1}, b]$ and

$$\bigcup_{i=1}^{n-1} f^i([w-\varepsilon, w+\varepsilon]) \subset [a, w-\varepsilon] - S \tag{5.4}$$

Then $f^n|_{[w,w+\varepsilon]}$ is increasing since $\mathcal{O}(w,f)$ is in an even state. Thus $f^n([w,w+\varepsilon]) \subset [w,b]$, and hence $[w,w+\varepsilon] \subset X_0$.

Let X be the connected component of X_0 containing $[w,w+\varepsilon]$. Then there is $z \in [w+\varepsilon, b]$ such that $X = [w,z]$. Obviously, one has $z = b \in S$ or $f^n(z) = w$. We claim that the following two equations hold:

$$\bigcup_{i=1}^{n-1} f^i(X) \subset [a, w-\varepsilon]. \tag{5.5}$$

$$f^i(X) \cap f^j(X) = \emptyset, \quad \text{for all} \quad 0 \leq i < j < n. \tag{5.6}$$

In fact, if (5.5) does not hold, then there will be $x \in X$ and $j \in \mathbb{Z}_{n-1}$ such that $f^j(x) = w - \varepsilon$. This implies $f^n(x) = f^{n-j}(w-\varepsilon) \in [a, w-\varepsilon]$, which is a contradiction to (5.3). Similarly, if (5.6) does not hold, then there will be $x, y \in X$ and $i, j \in \{0, 1, \ldots, n-1\}$ with $i < j$ such that $f^i(x) = f^j(y)$. This implies $f^n(y) = f^{n-j}(f^j(y)) = f^{n-j}(f^i(x)) = f^{n-j+i}(x) \in [a, w-\varepsilon]$, which is also a contradiction to (5.3). Thus (5.5) and (5.6) must hold.

Let $Y = X \cap P_n(f)$. Then Y is closed and $w \in Y$. Let $v = \max Y$ and $\mathcal{O}(v,f) = V = \{v_1 < v_2 < \ldots < v_n\}$. It follows from (5.5) that $v = v_n$ and

$$f^n(x) \neq x \quad \text{for all} \quad x \in (v, z]. \tag{5.7}$$

By (5.6) one has (5.2). Hence V and W have the same pattern as θ. Since $f|_S$ has no periodic orbit of pattern θ, $V \cap S = \emptyset$.

We now prove that V is in an odd state. In fact, if V is in an even state, putting

$$T = \{x \in [v,z] : f^n(x) \in S\},$$

then T is closed and $T \subset (v,z] \subset X$. If $T = \emptyset$, then $z < b$ and there exists $\delta \in (0, b-z]$ such that $f^n([v, z+\delta]) \cap S = \emptyset$, which implies $\cup_{i=0}^{n-1} f^i([v,z+\delta]) \cap S = \emptyset$. From this it follows that $f^n|_{[v,z+\delta]}$ is increasing, $f^n([v,z+\delta]) \subset [v,b] \subset [w,b]$ and hence $[w, z+\delta] \subset X$. But this contradicts with $X = [w,z]$. If $T \neq \emptyset$, writing $t = \min T$ and $b_1 = \min(S \cap [v,b])$, then $b_1 \geq t > v$. Since $f^n([v,t)) \cap S = \emptyset$, one has $\cup_{i=0}^{n-1} f^i([v,t)) \cap S = \emptyset$. Thus $f^n|_{[v,t]}$ is increasing and $f^n(t) = b_1$. Noting $f|_S$ has no periodic orbit of pattern θ, by (5.6) one has $b_1 > t$. This with (5.7) yields $f^n(z) > z$, which still contradicts with that $z = b$ or $f^n(z) = w$.

Therefore, V must be in an odd state and hence $v > w$. Since

$$\bigcup_{i=1}^{n} [w_i; v_i] \subset \bigcup_{j=0}^{n-1} f^j([w,v]) \subset \bigcup_{j=0}^{n-1} f^j(X) \subset \bigcup_{j=0}^{n-1} f^j(X_0),$$

by (5.3) one can obtain (5.1). $\square$

Theorem 5.2. *Let (X,ψ) be a compact line system, and $\theta \in \mathbf{C}$ be a pattern of periodic orbit. Then $\psi \Rightarrow \theta$ if and only if there exists a monotonic extension of ψ which has a periodic orbit of pattern θ.*

Proof. It is enough to show the sufficiency. If ψ itself has a periodic orbit of pattern θ, then one has $\psi \Rightarrow \theta$ immediately. Now assume that ψ has no periodic orbit of pattern θ, but there is a monotonic extension f of ψ which has a periodic orbit W of pattern θ. Then, by Lemma 5.1, f has a periodic orbit V of pattern θ in an odd state. Evidently, f with V satisfies the conditions (C.1)$-$(C.4) in Theorem 4.12. Thus one has $\psi \Rightarrow \theta$. □

5.2. **Non-periodic minimal patterns.** Now we study the non-periodic minimal patterns. Note that even if $(X,\psi) \in \Psi$ is minimal, not every element in $[(X,\psi)]$ is minimal. And generally we do not have a similar theorem like Theorem 5.2 for a minimal pattern. But if we use the definition by Bobok [13], we can say more. Firstly let's recall some definitions.

Let $(X,\psi) \in \Psi$. Say a system (X,ψ) is **minimal** if it is compact and every point in X has a dense orbit. It is easy to see that a compact system (X,ψ) is minimal if and only if it has no proper nonempty closed invariant subset. A point x is said to be **minimal** if its orbit closure is a minimal system. It is well known that a point x is minimal if and only if its recurrent time is syndetic, i.e. for any neighborhood U of x, the set $N(x,U) = \{n \in \mathbb{N} : f^n(x) \in U\}$ has a bounded gap (see, for example, [21]). Obviously, each periodic orbit is minimal. And it is easy to verify that if x is minimal under f, then it is also minimal under f^n for any $n \in \mathbb{N}$.

Denote $\mathcal{M} = \{(X,\psi) \in \Psi : (X,\psi) \text{ is minimal }\}$. For $(X,\psi),(Y,\xi) \in \mathcal{M}$, one says that (X,ψ) is **B-equivalent** to (Y,ξ), denoted by $(X,\psi) \approx_B (Y,\xi)$, if the map $h : \mathcal{O}(\min X,\psi) \to \mathcal{O}(\min Y,\xi)$ defined by $h(\psi^n(\min X)) = \xi^n(\min Y)$ for all $n \geq 0$ is an order-preserving bijection. For $(X,\psi) \in \mathcal{M}$, write

$$[(X,\psi)]_B = \{(Y,\xi) \in \mathcal{M} : (Y,\xi) \approx_B (X,\psi)\}.$$

Lemma 5.3. *Let (X,f) and (Y,g) be compact line systems. Suppose that (X,f) is minimal but not periodic, and $x = \min X$. If there exists $y \in Y$ such that the map $h : \mathcal{O}(x,f) \to \mathcal{O}(y,g)$ defined by $h(f^n(x)) = g^n(y)$ for all $n \in \mathbb{Z}_+$ is an order-preserving bijection, then there exists a minimal set Y' of g such that $(X,f) \approx_B (Y',g|_{Y'})$.*

Proof. Let $y = h(x)$. Since (X,f) is minimal but not periodic, there is some strictly decreasing sequence $\{f^{n_i}(x)\}_{i=1}^{\infty}$ such that $x = \lim_{i\to\infty} f^{n_i}(x)$. For each $n \in \mathbb{N}$, denote $f^n(x)$ and $g^n(y)$ by x_n and y_n respectively. Since h is order-preserving, $\{y_{n_i}\}_{i=1}^{\infty}$ is also strictly decreasing and hence converges to some point $y' \in \overline{Y} = Y$. Let $Y' = \overline{\mathcal{O}(y',g)}$. Define $h' : \mathcal{O}(x,f) \to \mathcal{O}(y',g)$ by $h'(f^n(x)) = g^n(y')$ for any $n \in \mathbb{N}$.

Firstly, we show $\mathcal{O}(y',g)$ is infinite. Let $k,j,n \in \mathbb{N}$ such that $f^k(x) < f^n(x) < f^j(x)$. By continuity of f^n, there exists an $i_0 \in \mathbb{N}$ such that $f^k(x) < f^n(x_{n_i}) < f^j(x)$ for any $i \geq i_0$. Since h is order-preserving on $\mathcal{O}(x,f)$, $g^k(y) < g^n(y_{n_i}) < g^j(y)$ for any $i \geq i_0$. Letting i tend to ∞, one has $g^n(y') \in [g^k(y),g^j(y)]$. From this it is easy to see that $\mathcal{O}(y',g)$ is infinite.

Consider any $j, k \in \mathbb{N}$. If $f^k(x) < f^j(x)$, then, since f is continuous, there exists an $i_0 \in \mathbb{N}$ such that $f^k(x_{n_i}) < f^j(x_{n_i})$ for any $i \geq i_0$. But h is order-preserving on $\mathcal{O}(x, f)$, hence $g^k(y_{n_i}) < g^j(y_{n_i})$ for any $i \geq i_0$. Taking limit, one has $g^k(y') \leq g^j(y')$. Since $\mathcal{O}(y', g)$ is infinite, one get $g^k(y') < g^j(y')$, i.e. $h'(f^k(x)) < h'(f^j(x))$. This means that h' is order-preserving, and it follows from $x = \min(\mathcal{O}(x, f))$ that $y' = \min(\mathcal{O}(y', g))$. Thus $y' = \min Y'$.

Now we show that y' is a minimal point. It is well known that a point is minimal if and only if the sequence of the times at which this point returns to any given neighborhood is syndetic, i.e. has a bounded gap. So it suffices to show that for any $\varepsilon > 0$, the set $N(y', [y', y' + \varepsilon)) \equiv \{k \in \mathbb{N} : g^k(y') \in [y', y' + \varepsilon)\}$ is syndetic. Since $y' = \lim_{i\to\infty} y_{n_i}$, there is some $j \in \mathbb{N}$ such that $y_{n_j} = g^{n_j}(y) \in [y', y' + \varepsilon)$. Now consider $[x, x_{n_j})$. For any $k \in N(x, [x, x_{n_j}))$, one has $f^k(x) \in [x, x_{n_j})$. Similar to the analysis above, one can show $g^k(y') \in [y', y_{n_j}) \subset [y', y' + \varepsilon)$. That means $N(x, [x, x_{n_j})) \subset N(y', [y', y' + \varepsilon))$. As x is minimal, $N(x, [x, x_{n_j}))$ and hence $N(y', [y', y' + \varepsilon))$ is syndetic. So y' is minimal and $(Y', g|_{Y'})$ is a minimal system. □

Definition 5.4. Let (W, φ) and (X, ψ) be line systems. Suppose that (X, ψ) is minimal but not periodic. We say that $[(W, \varphi)]$ **forces** $[(X, \psi)]_B$ and write $[(W, \varphi)] \Rightarrow [(X, \psi)]_B$ if any interval map exhibiting (W, φ) has a minimal set which is B-equivalent to (X, ψ).

Theorem 5.5. *Let (W, φ) and (X, ψ) be line systems. Suppose that (X, ψ) is minimal but not periodic, $x = \min X$, $X' = \mathcal{O}(x, \psi)$, and $\psi' = \psi|_{X'} : X' \to X'$. If $[(W, \varphi)] \Rightarrow [(X', \psi')]$, then $[(W, \varphi)] \Rightarrow [(X, \psi)]_B$.*

Proof. Let g be an interval map which exhibits (W, φ). Then g exhibits (X', ψ') since $[(W, \varphi)] \Rightarrow [(X', \psi')]$. Thus there exist an invariant set Y_0 of g and an order-preserving bijection $h : X' \to Y_0$ such that $h\psi' = gh$. Let $y = h(x)$ and $Y_0 = \mathcal{O}(y, g)$. By Lemma 5.3, there exists a minimal set Y' of g such that $(X, f) \approx_B (Y', g|_{Y'})$. This means that $[(W, \varphi)] \Rightarrow [(X, \psi)]_B$. □

For any $(X, \psi) \in \Psi$ and any $x \in X$, one has $[(X, \psi)] \Rightarrow [(\mathcal{O}(x, \psi), \psi|_{\mathcal{O}(x,\psi)})]$. Hence, from Theorem 5.5 we obtain

Corollary 5.6. *Let (W, φ) and (X, ψ) be line systems. Suppose that (X, ψ) is minimal but not periodic. If $[(W, \varphi)] \Rightarrow [(X, \psi)]$, then $[(W, \varphi)] \Rightarrow [(X, \psi)]_B$.*

Lemma 5.7. *Let (X, ψ) be a minimal line system but not periodic, and $x \in X$. Then $(\mathcal{O}(x, \psi), \psi|_{\mathcal{O}(x,\psi)})$ has no companionate orbits.*

Proof. Write $x_n = \psi^n(x)$ for all $n \in \mathbb{Z}_+$. If there exist $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$ such that $\mathcal{O}(x_n, \psi)$ and $\mathcal{O}(x_{n+k}, \psi)$ are a pair of companionate orbits, then, by (ii) of Definition 4.13, $(x_n, x_{n+k}, x_{n+2k}, x_{n+3k}, \cdots)$ is a strictly monotonic sequence, and x_n is a minimal point of neither ψ^k nor ψ. But this will lead to a contradiction. Thus Lemma 5.7 is true. □

Theorem 5.8. *Let (W, φ) be a compact line system, and (X, ψ) be a minimal line system but not periodic. If there exists a monotonic extension f of (W, φ) exhibiting (X, ψ), then $[(W, \varphi)] \Rightarrow [(X, \psi)]_B$.*

Proof. Let $x = \min X$, $X' = \mathcal{O}(x, \psi)$, and $\psi' = \psi|_{X'} : X' \to X'$. Then f also exhibits (X', ψ'). By Lemma 5.7, (X', ψ') has no companionate orbits. It follows from Theorem 4.14 that $[(W,\varphi)] \Rightarrow [(X',\psi')]$, which with Theorem 5.5 implies that $[(W,\varphi)] \Rightarrow [(X,\psi)]_B$. □

6. Fissions of periodic orbits

In this section, as applications of the results built in Section 4, we discuss a kind of invariant sets, which can be obtained by repeatedly 2-fissioning periodic orbits. Before that, we give a generalization of Proposition 3.11.

6.1. **A generalization of Proposition 3.11.**

Definition 6.1. Let $\theta \in \mathbf{C}_n$, $n \geq 2$ and X be a nonempty subset of $\mathbb{R}$. A continuous map $\varphi : X \to X$ is said to be **θ-separable** if there exist open intervals $J_1 < J_2 < \ldots < J_n$ such that $X \subset \bigcup_{i=1}^n J_n$ and $\varphi(X \cap J_k) \subset X \cap J_{\theta(k)}$ for $k = 1, 2, \ldots, n$.

Let $f \in C^0(I), x \in I = [a,b]$ and $\theta \in \mathbf{C}_n$. For $m \geq n \geq 2, \mathcal{O}_m(x,f) = \{f^i(x) : i = 0,1,\ldots,m\}$ is said to be **θ-separable** if there exist intervals $J_1 < J_2 < \ldots < J_n$ such that $\mathcal{O}_m(x,f) \subset \cup_{i=1}^n J_i$ and $f(\mathcal{O}_{m-1}(x,f) \cap J_k) \subset \mathcal{O}_m(x,f) \cap J_{\theta(k)}$ for $k = 1,\ldots,n$.

Obviously, θ itself is θ-separable. It is easy to see that every doubling of θ is θ-separable.

Proposition 6.2. *Let $\theta \in \mathbf{C}_n, n \geq 2$, and $(X,\psi) \in \Psi$. If ψ is θ-separable and X is compact, then $\psi \Rightarrow \theta$.*

Proof. Consider any $f \in C^0(I)$. Assume f has an invariant set S with the pattern of ψ. Since X is compact, by Definitions 4.1 and 6.1 it is easy to verify that there exist closed intervals $J_1 < J_2 < \ldots < J_n$ such that $\cup_{i=1}^n \partial J_i \subset S \subset \cup_{i=1}^n J_i$ and $f(S \cap J_i) \subset S \cap J_{\theta(i)}$ $(i = 1,\ldots,n)$. Let g be a normal reduction of f relative to $\overline{S}$. Then $g(J_i) \subset J_{\theta(i)}, (i = 1,\ldots,n)$. Thus there exists $x \in J_1 \cap P_n(g)$ such that $\mathcal{O}(x,f) = \mathcal{O}(x,g)$ has the pattern θ. This implies that $\psi \Rightarrow \theta$. □

The following theorem with Proposition 6.2 is a generalization of Proposition 3.11.

Theorem 6.3. *Let $(X,\psi),(Y,\xi) \in \Psi$ and $\theta \in \mathbf{C}$. Suppose ψ is θ-separable and there exists $y \in Y \cap \{\inf Y, \sup Y\}$ such that $\mathcal{O}(y,\xi)$ is not θ-separable. Then $\theta \Rightarrow \xi$ if and only if $\psi \Rightarrow \xi$.*

Proof. By Proposition 6.2, we need only to verify the sufficiency. Without loss of generality, we may assume $y = \inf Y \in Y$. Let $n \geq 2$ be the period of θ. Suppose $f \in C^0(I)$ has a periodic orbit $W = \{w_1 < w_2 < \ldots < w_n\}$ of pattern θ. Now we show that f has an invariant set with the pattern of ξ. Since $\mathcal{O}(y,\xi)$ is not θ-separable, there exists $m \geq 2n$ such that $\mathcal{O}_m(y,\xi)$ is not θ-separable. Let $g \in C^0(I)$ be defined by

$$g(x) = f(\max\{w_1, \min\{w_n, x\}\}), \quad \text{for any} \quad x \in I.$$

For any $r > 0$, let

$$B(g,r) = \{\varphi \in C^0(I) : |\varphi(x) - g(x)| \leq r \ \text{ for any } x \in I\},$$

and

$$B(W,r) = \{x \in I : x \leq w_1, \text{ or } x \geq w_n, \text{ or } |x - w_i| \leq r \text{ for some } i \in \mathbb{Z}_n\}.$$

By continuity, there exist $\varepsilon \in (0, \min\{w_i - w_{i-1} : i = 2, 3, \ldots, n\}/3]$ and $\delta \in (0, \varepsilon/5]$ such that $\mathcal{O}_{m+n}(z, \varphi)$ is θ-separable for any $\varphi \in B(g, \varepsilon)$ and any $z \in B(W, \varepsilon)$, and $\mathcal{O}_n(u, \zeta) \cap [w_1 - \varepsilon, w_1 + \varepsilon] \neq \emptyset$ for any $\zeta \in B(g, 2\delta)$ and any $u \in B(W, 2\delta)$. Noting ψ is θ-separable, we can construct a map $\eta \in B(g, 2\delta)$ such that $\eta(B(W, , \delta)) \subset B(W, \delta), \eta|_{(I-B(W,2\delta))} = g|_{(I-B(W,2\delta))}$, and $\eta|_{B(W,\delta)}$ has an invariant set with the pattern of ψ. Since $\psi \Rightarrow \xi, \eta$ has an invariant set V with the pattern of ξ. Let $v = \min V$. Then $\mathcal{O}_m(v, \eta)$ is not θ-separable. Thus $v > w_1 + \varepsilon$, and hence $V \cap B(W, 2\delta) = \emptyset$. This implies $\eta|_V = g|_V = f|_V$, and hence f has an invariant set with the pattern of ξ. □

6.2. **Fissions of periodic orbits.**

Definition 6.4. Let $I = [a, b]$ and $f \in C^0(I)$. Suppose $W = \{w_1 < w_2 < \ldots < w_n\}$ and $V = \{v_1 < v_2 < \ldots < v_{2n}\}$ are periodic orbits of f with pattern γ and η respectively. V (or $f|_V$) is called a 2-**fission** of W (or $f|_W$) if the pattern η is a doubling of γ and $w_i \in (v_{2i-1}, v_{2i})$ for all $i \in \mathbb{Z}_n$.

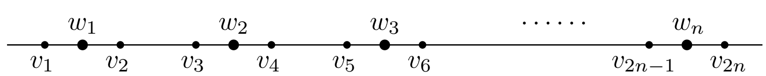

Figure 5

Lemma 6.5. *Let $I = [a, b]$, $f \in C^0(I)$ and $W = \{w_1 < w_2 < \ldots < w_n\}$ be a periodic orbit of f. Suppose one of the following two conditions holds:*

(i) *There exists $z \in I$ such that $f^n(z) < w_n < z < f^{2n}(z)$, and $f^{2n}([w_n, z]) \subset [w_n, b]$.*

(ii) *W is in an odd state under f and there exists a nonempty closed invariant set S of f such that $W \subset L(S) - S$ and f is monotonic on every connected component of $L(S) - S$.*

Then there exists a 2-fission $V = \{v_1 < v_2 < \ldots < v_{2n}\}$ of W such that

$$\mathcal{O}(x, f) \not\subset (w_1, w_n) \quad \text{for any} \quad x \in \bigcup_{i=1}^{n} [v_{2i-1}, v_{2i}]. \tag{6.1}$$

Proof. (1) We first assume the condition (i) holds. Let $Y = \{x \in [z, b] : f^{2n}(x) = x\}$. Then Y is nonempty and closed, since $f^{2n}(z) \geq z$ and $f^{2n}(b) \leq b$. Let $v = \min Y$ and $w = w_n$. Then

$$f^{2n}([z, v]) \subset [z, b] \quad \text{and} \quad f^{2n}([w, v]) \subset [w, b]. \tag{6.2}$$

Claim. For any $i \in \mathbb{Z}_{2n-1}, f^i(v) < w$.

Proof of Claim. If $f^i(v) \geq w$ for some $i \in \mathbb{Z}_{2n-1} - \{n\}$, then it follows from $f^i(w) < w$ that there exists $y \in (w, v]$ such that $f^i(y) = w$ and $f^{2n}(y) = f^{2n-i}(w) < w$, which contradicts with (6.2). If $f^n(v) \geq w$, then it follows from $f^n(z) < w$ that there exists $y \in (z, v]$ satisfying $f^n(y) = w$ and $f^{2n}(y) = f^n(w) = w < z$, which also contradicts (6.2). Thus Claim holds.

If the periodic orbit $\mathcal{O}(v, f)$ is not a 2-fission of W, then there exist i and $j \in \mathbb{Z}_{2n}$ with $i \neq j$ such that $f^i(v) \in (f^j(w); f^j(v)]$. Thus there is $y \in (w, v]$ such that $f^j(y) = f^i(v)$. By Claim, we have $f^{2n}(y) = f^{2n-j+i}(v) < w$. But this contradicts (6.2). Hence $\mathcal{O}(v, f)$ must be a 2-fission of W. Particularly, the period of $\mathcal{O}(v, f)$ must be $2n$.

Suppose $V = \{v_1 < v_2 < \ldots < v_{2n}\} = \mathcal{O}(v, f)$. Then $v = v_{2n}$. For any $j \in \mathbb{Z}_{2n}$ and any $x \in [f^j(w); f^j(v)]$, there exists $y \in [w, v]$ such that $f^j(y) = x$ and $f^{2n-j}(x) = f^{2n}(y) \in [w, b]$. Thus (6.1) holds.

(2) We now assume that the condition (ii) holds. Suppose (y_n, z_n) is the connected component of $L(S) - S$ containing $w(= w_n)$. let

$$X = \{x \in (y_n, z_n) : f^i(x) \notin S \ \text{ for all } \ i \in \mathbb{Z}_{2n}\}.$$

Then X is an open set and $w \in X \subset (y_n, z_n)$. Let X_0 be the connected component of X containing w. Let $z = \sup X_0$. For every $i \in \mathbb{Z}_{2n}$, since $f^i([w, z))$ is in a connected component of $L(S) - S$, $f|_{f^i([w,z))}$ is monotonic. Noting W is in an odd state, we see that $f^n|_{[w,z]}$ is decreasing. Since $f(S) \subset S$, we must have $y_n \le f^n(z) < w < z \le z_n = f^{2n}(z)$ and $f^{2n}([w, z]) = [w, z_n] \subset [w, b]$. Hence the condition (i) holds. The proof is completed. □

Definition 6.6. Let $(X, \xi) \in \Psi$, X_0 be a periodic orbit of ξ with period $m \ge 1$, $\xi_0 = \xi|_{X_0}$ and $n \in \mathbb{N}$.

(1) ξ is called a $(1, 2, 4, \ldots, 2^n)$**-fission** of ξ_0 if there exist periodic orbits X_0, X_1, …, X_n of ξ such that $X = \bigcup_{i=0}^n X_i$ and

(i) for every $i \in \mathbb{Z}_n$, the period of X_i is $2^i m$, and $\xi|_{X_i}$ is a 2-fission of $\xi|_{X_{i-1}}$;

(ii) If $n \ge 2$, then $[x; \xi^{2^{i-1}m}(x)] \cap (\bigcup_{j=0}^{i-2} X_j) = \emptyset$ for any $i \in \{2, \ldots, n\}$ and any $x \in X_i$.

(2) ξ is called a 2^∞**-fission** of ξ_0 if there exist periodic orbits $X_0, X_1, X_2, \ldots$ of ξ such that $X = \bigcup_{i=0}^\infty X_i$ and for any $i \in \mathbb{N}$, $\xi|_{\bigcup_{j=0}^i X_j}$ is a $(1, 2, 4, \ldots, 2^i)$-fission of ξ.

Example 6.7. Let $X = \mathbb{Z}_7, X_0 = \{3\}, X_1 = \{2, 6\}, X_2 = \{1, 4, 5, 7\}$. Define $\xi : X \to X$ by $\xi(3) = 3, \xi(2) = 6, \xi(6) = 2, \xi(1) = 5, \xi(4) = 7, \xi(5) = 4, \xi(7) = 1$ (See Figure 6). Then $\xi|_{X_i}$ is a 2-fission of $\xi|_{X_{i-1}} (i = 1, 2)$. But ξ is not a $(1, 2, 4)$-fission of $\xi|_{X_0}$. From this example we see that the condition (i) in Definition 6.6 does not imply the condition (ii).

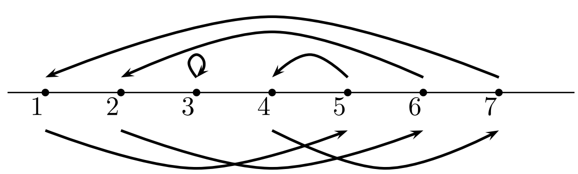

Figure 6

Proposition 6.8. *Let $\xi : X \to X$ be a 2^∞-fission of ξ_0 and $X_0, X_1, X_2, \ldots$ be as in Definition 6.6. Suppose the pattern of ξ_0 is $\theta \in \mathbf{C}_m$ $m \ge 1$. Then*

(i) $\xi|_{(X - \bigcup_{i=0}^{n-1} X_i)}$ *is also a 2^∞-fission of $\xi|_{X_n}, (n = 1, 2, \ldots)$;*

(ii) *ξ is θ-separable;*

(iii) *every point in X is an isolated point of X.*

Proof. By Definitions 6.6 and 6.1, (i) and (ii) are obvious. We now check (iii). By (ii), we may consider only the case $m = 1$. Suppose the unique point in X_0 is x_0. By (i), it suffices to show that x_0 is an isolated point of X. Suppose $X_1 = \{y_1 < y_2\}, X_2 = \{v_1 < v_2 < v_3 < v_4\}$. Then $v_1 < y_1 < v_2 < x_0 < v_3 < y_2 < v_4$ and $\xi(v_2) \in \{v_3, v_4\}$. If $\xi(v_2) = v_4$, then $\xi(X \cap [y_1, x_0)) \subset [y_2, \sup X)$. By the continuity of ξ at x_0, we have $\sup(X \cap [y_1, x_0)) < x_0$, and hence $\inf(X \cap (x_0, y_2]) > x_0$. Thus x_0 is an isolated point of X. If $\xi(v_2) = v_3$, then $\xi(v_3) = v_1$. By an analogous argument, x_0 is still an isolated point of X. $\square$

For any $\theta \in \mathbf{C}$, let $\mathbf{D}_0(\theta) = \{\theta\}$. For $k = 1, 2, 3, \ldots$, let

$$\mathbf{D}_k(\theta) = \{\eta \in \mathbf{C} : \text{ there is } \xi \in \mathbf{D}_{k-1}(\theta) \text{ such that } \eta \text{ is a doubling of } \xi\},$$

and let $\mathbf{D}_k^*(\theta) = \bigcup_{i=0}^k \mathbf{D}_i(\theta)$. Write $\mathbf{D}_\infty(\theta) = \bigcup_{i=1}^\infty \mathbf{D}_i(\theta)$. Let $\mathbf{C}(\psi) = \{\theta \in \mathbf{C}$: there is some periodic orbit of ψ equivalent to $\theta\}$, where $(X, \psi) \in \Psi$.

Theorem 6.9. *Let $X \subset \mathbb{R}$ be compact, $(X, \psi) \in \Psi$ and $\theta \in \mathbf{C}$.*

(i) *Suppose $\mathbf{C}(\psi) \cap \mathbf{D}_{n-1}^*(\theta) = \emptyset$, for some $n \in \mathbb{N}$. Then $\psi \Rightarrow \theta$ if and only if there exists a $(1, 2, 4, \ldots, 2^n)$-fission ξ_n of θ such that $\psi \Rightarrow \xi_n$.*

(ii) *Suppose $\mathbf{C}(\psi) \cap \mathbf{D}_\infty(\theta) = \emptyset$. Then $\psi \Rightarrow \theta$ if and only if there exists a 2^∞-fission ξ of θ such that $\psi \Rightarrow \xi$.*

Proof. The sufficiency is evident. We now prove the necessity. Suppose $\theta \in \mathbf{C}_m$ for some $m \geq 1$. Write $m_i = 2^i m$. Let $I = [a, b] = L[X], U = I - X$ and let $f \in C^0(I)$ be the linear extension of ψ. Assume $\psi \Rightarrow \theta$. Then f has a periodic orbit W_0 of pattern θ. Since $\mathbf{C}(\psi) \cap \mathbf{D}_0(\theta) = \emptyset$, $W_0 \subset U$. By Lemma 5.1, we may assume W_0 is in an odd state. By Lemma 6.5, f has a 2-fission V_1 of W_0.

We now assume that, for some $k \geq 1, f$ has periodic orbits $W_0, W_1, \ldots, W_{k-1}$ and V_k satisfying the following four conditions:

(a) $f|_{W_0}$ has the same pattern as θ;

(b) for $1 \leq i \leq k-1$, $f|_{W_i}$ is a 2-fission of $f|_{W_{i-1}}$, and $f|_{V_k}$ is a 2-fission of $f|_{W_{k-1}}$;

(c) $f|_{(W_0 \cup W_1 \cup \ldots \cup W_{k-1} \cup V_k)}$ is a $(1, 2, 4, \ldots, 2^k)$-fission of $f|_{W_0}$;

(d) for $0 \leq i \leq k-1$, $W_i \subset U$ and $f|_{W_i}$ is in an odd state.

Since $\mathbf{C}(\psi) \cap \mathbf{D}_k(\theta) = \emptyset$, we have $V_k \subset U$. Let $V_k = \{v_{k1} < v_{k2} < v_{k3} < \ldots < v_{km_k}\}$. If $f|_{V_k}$ is in an even state, then by Lemma 5.1, f has a periodic orbit $W_k = \{w_{k1} < w_{k2} < \ldots < w_{km_k}\} \subset U$ satisfying the following two conditions:

(e) $f|_{W_k}$ is in an odd state and has the same pattern as $f|_{V_k}$;

(f) $\mathcal{O}(x, f) \not\subset [a, v_{km_k})$ for any $x \in \bigcup_{i=1}^{m_k} [v_{ki}; w_{ki}]$, and $[v_{ki}, w_{ki}] \cap [v_{kj}, w_{kj}] = \emptyset$ for $1 \leq i < j \leq m_k$.

It follows from (a)$-$(f) that $f|_{W_k}$ is also a 2-fission of $f|_{W_{k-1}}$, $(\cup_{j=1}^{m_k/2} [w_{k,2j-1}, w_{k,2j}]) \cap (\cup_{i=0}^{k-2} W_i) = \emptyset$, and $f|_{\cup_{i=0}^k W_i}$ is also a $(1, 2, 4, \ldots, 2^k)$-fission of $f|_{W_0}$. If $f|_{V_k}$ is in an odd state, then we put $W_k = V_k$ and $w_{kj} = v_{kj}$, $j = 1, 2, \ldots, m_k$. By Lemma 6.5, f has a periodic orbit $V_{k+1} = \{v_{k+1,1} < v_{k+1,2} < \ldots < v_{k+1,m_{k+1}}\}$ satisfying

(g) $f|_{V_{k+1}}$ is a 2-fission of $f|_{W_k}$;

(h) $\mathcal{O}(x, f) \not\subset (w_{k1}, w_{km_k})$ for any $x \in \bigcup_{i=1}^{m_k} [v_{k+1,2i-1}, v_{k+1,2i}]$.

From (a)$-$(h) we see that $(\bigcup_{i=1}^{m_k} [v_{k+1,2i-1}, v_{k+1,2i}]) \bigcap (\bigcup_{j=0}^{k-1} W_j) = \emptyset$, and $f|_{(W_0 \cup W_1 \cup \ldots \cup W_k \cup V_{k+1})}$ is a $(1, 2, 4, \ldots, 2^{k+1})$-fission of $f|_{W_0}$.

By induction, we obtain the following

Claim. (I) If $\mathbf{C}(\psi) \cap \mathbf{D}_{n-1}^*(\theta) = \emptyset$, then f has an invariant set $Y_n = W_0 \cup W_1 \cup \ldots \cup W_{n-1} \cup V_n$ such that $f|_{Y_n}$ is a $(1, 2, 4, \ldots, 2^n)$-fission of $f|_{W_0}$;

(II) If $\mathbf{C}(\psi) \cap \mathbf{D}_\infty(\theta) = \emptyset$, then f has an invariant set $Y = \cup_{j=0}^{\infty} W_j$ such that $f|_Y$ is a 2^∞-fission of $f|_{W_0}$.

By this claim and Theorem 4.14 one completes the proof of Theorem 6.9. □

7. Entropies of patterns of compact line systems

Firstly recall the definition of the topological entropy. Let (X, f) be a compact system. For $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset W of X is called an (f, ε, n)**- spanning set** of X if for any $x \in X$ there is $y \in W$ such that $d(f^i x, f^i y) < \varepsilon$ for $1 \leq i \leq n$. Let $Span(f, \varepsilon, n)$ denote the smallest cardinality of any (f, ε, n)-spanning set of X. Then the **topological entropy of** (X, f) is defined by

$$h(X, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Span(f, \varepsilon, n).$$

See [1, 9, 11] etc. for different definitions and more information about the topological entropy.

Definition 7.1. Let (X, ψ) be a line system, and $I = L(X)$. Define

$$h^*(X, \psi) = \inf\{h(I, f) : f \in C^0(I) \text{ and } f|_X = \psi\},$$

$$h^*[(X, \psi)] = \inf\{h^*(Y, \xi) : (Y, \xi) \in [(X, \psi)]\}.$$

One says that $h^*[(X, \psi)]$ is the **topological entropy of the pattern** $[(X, \psi)]$.

When (X, ψ) is a periodic orbit, $h^*[(X, \psi)] = h^*(X, \psi)$ is extensively studied by lots of authors and there are lots of interesting results (for example, see [1, 7, 8, 9, 11]). It is shown that, for any periodic orbit (X, ψ), $h^*(X, \psi)$ is the entropy of its linear extension. The following lemma is also well known, see [1] .

Lemma 7.2. *For any interval map $f : I \to I$, it holds that*

$$h(I, f) = \sup\{h^*(P, f|_P) : P \text{ is a periodic orbit of } f\}.$$

For an interval map, the topological entropy is also closely related to its minimal subsets, which is discussed in [14]. We now give a theorem, which is a generalization of the corresponding results on patterns of periodic orbit.

Theorem 7.3. *Let (X, ψ) be a compact line system, $I = L(X)$, and f be a monotonic extension of ψ. Then $h^*[(X, \psi)] = h^*(X, \psi) = h(I, f)$.*

Proof. It follows from Definition 7.1 that $h^*[(X, \psi)] \leq h^*(X, \psi) \leq h(I, f)$. Hence, it suffices to show $h^*[(X, \psi)] \geq h(I, f)$. Consider any given real number $r < h(I, f)$ and any interval map $g : J \to J$ exhibiting (X, ψ). By Lemma 7.2, there is a periodic orbit P of f such that $h^*(P, f|_P) > r$. By Theorem 5.2, g exhibits $(P, f|_P)$, and hence, by Lemma 7.2 again, one has $h(J, g) \geq h^*(P, f|_P) > r$. This means that $h^*[(X, \psi)] \geq h(I, f)$. □

From Theorem 7.3 we obtain the following corollary at once.

Corollary 7.4. *(1) Let (X, ψ) be a compact line system, $I = L(X)$, and f and g be two monotonic extensions of ψ. Then $h(I, f) = h(I, g)$.*

(2) Let (X, ψ) and (Y, ξ) be two compact line systems which have the same pattern. Then $h^*(X, \psi) = h^*(Y, \xi)$.

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References

1. Ll. Alseda, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second ed., World Scientific Publishing Co.Inc., River Edge, NJ, 2000.
2. S. Baldwin, *Generalizations of a theorem of Sharkovskiĭ on orbits of continuous real-valued functions*, Discrete Math., **67**(1987), 111–127.
3. C. Bernhardt, *The ordering on permutations induced by continuous maps of the real line*, Ergodic Theory Dynamical Systems, **7**(1987), 155–160.
4. C. Bernhardt, E. Coven, M. Misiurewicz and I. Mulvey, *Comparing periodic orbits of maps of the interval*, Trans. Amer. Math. Soc., **333**(1992), 701–707.
5. A.M. Blokh, *Rotation numbers, twists and a Sharkovskiĭ-Misiurewicz-pattern ordering for patterns on the interval*, Ergodic Theory Dynam. Systems 15 (1995), no. 1, 1–14.
6. A.M Blokh and M. Misiurewicz, *New order for periodic orbits of interval maps*, Ergodic Theory Dynam. Systems 17 (1997), no. 3, 565–574.
7. A.M Blokh and M. Misiurewicz, *Entropy of twist interval maps*, Israel J. Math. 102 (1997), 61–99.
8. A.M Blokh and M. Misiurewicz, *Entropy and over-rotation numbers for interval maps*, Tr. Mat. Inst. Steklova 216 (1997), Din. Sist. i Smezhnye Vopr., 236–242; translation in Proc. Steklov Inst. Math. 1997, no. 1 (216), 229–235
9. L. S. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, 1513. Springer-Verlag, Berlin, 1992.
10. L. Block and E. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. math. Soc., **300**(1987), 297–306.
11. L. Block, J. Guckenheimer, M. Misiurewicz and L. S. Young, *Periodic points and topological entropy*, Lecture Notes in Math., Vol.**819**, Springer, Berlin, 1980, pp. 18–34.
12. L. Block and D. Hart, *Orbit types for maps of the interval*, Ergodic Theory Dynamical Systems, **7**(1987), 161–164.
13. J. Bobok, *Forcing relation on minimal interval patterns*, Fund. Math. 169 (2001), no. 2, 161–173.
14. J. Bobok, *Strictly ergodic patterns and entropy for interval maps*, Acta Math. Univ. Comenian. (N.S.) 72 (2003), no. 1, 111–118.
15. J. Bobok and M. Kuchta, *X-minimal patterns and a generalization of Sharkovskiĭ's theorem*, Fund. Math. 156 (1998), no. 1, 33–66.
16. U. Burkart, *Interval mapping graphs and periodic points of continuous functions*, Journ. of Comb. Theory, **32**(1987), 57–68.
17. P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progress in Physics, vol. **I**, Birkhäuser, Basel, 1980.
18. J.-P. Delahaye, *Fonctions admettant des cycles d'ordre n'importe quelle puissance de 2 et aucun autre cycle*, C. R. Acad. Sci. Paris Sér. A-B, **291**(1980), A 323-325.
19. S. Elaydi, *On a converse of Sharkovsky's theorem*, Amer. Math. Monthly, **103**(1996), 386–392.
20. B. Gawel, *On the theorems of Sharkovskiĭ and Stefan on cycles*, Proc. Amer. Math. Soc., **107**(1989), 125–132.
21. W. Gottschalk and G. Hedlund, *Topological Dynamics*, Amer. Math. Soc., Providence (1955).
22. C. W. Ho and C. Morris, *A graph theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions*, Pacific J. Math., **96**(1981), 361–370.
23. I. Jungreis, *Some results on the Sharkovskiĭ partial ordering of permutations*, Trans. Amer. Math. Soc., **325**(1991), 319–344.
24. M. Misiurewicz and Z. Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. 94 (1991), no. 456.
25. A. N. Sharkovskiĭ, *Coexistence of cycles of a continuous mapping of the line into itself*, Ukrain. Math. Zh., **16**(1964), 61–71. (Russian)
26. P. Stefan, *A theorem of Sharkovskiĭ on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys., **54**(1977), 237–248.
27. P. D. Straffin, *Periodic points of continuous functions*, Math. Mag., **51**(1978). 99–105.

Department of Mathematics, Shantou University, Shantou, Guangdong, P. R. China
E-mail address: jhmai@stu.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China.
E-mail address: songshao@ustc.edu.cn