RELATIVELY WEAKLY MIXING MODELS FOR DYNAMICAL SYSTEMS

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Abstract. A classical result in ergodic theory says that there always exists a topological model for any factor map $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ of ergodic systems. That is, there is some topological factor map $\hat{\pi} : (\hat{X}, \hat{T}) \to (\hat{Y}, \hat{S})$ and invariant measures $\hat{\mu}$, $\hat{\nu}$ such that the diagram

$$\begin{array}{ccc} (X, \mathcal{X}, \mu, T) & \xrightarrow{\phi} & (\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\hat{\pi}} \\ (Y, \mathcal{Y}, \nu, S) & \xrightarrow{\psi} & (\hat{Y}, \hat{\mathcal{Y}}, \hat{\nu}, \hat{S}) \end{array}$$

is commutative, where ϕ and ψ are measure theoretical isomorphisms. In this paper, we show that one can require that in above result $\hat{\pi}$ is either weakly mixing or finite-to-one. Also we present some related questions in the paper.

1. Introduction

In ergodic theory a natural question is how to endow a given measurable system with a "nice" topological structure without destroying the original measurable structure. This kind of topological structure is called topological *model* or *realization*. To be precise, one has the following definition.

Definition 1.1. Let $(X, \mathcal{X}, \mu, T)$ be a measurable system. We say that the system $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ is a *topological model* (or just a *model*) for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, \hat{T})$ is a topological system, and there is some invariant measure $\hat{\mu}$ such that the systems $(X, \mathcal{X}, \mu, T)$ and $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ are measure theoretically isomorphic.

In general, one has the definition of topological model for a factor map between measurable systems.

Definition 1.2. Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map between two measurable systems. We say that $\hat{\pi} : (\hat{X}, \hat{T}) \to (\hat{Y}, \hat{S})$ is a *topological model* for $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ when $\hat{\pi}$ is a topological factor map and there exist

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invariant measures $\hat{\mu}$ and $\hat{\nu}$ such that the diagram

$$\begin{array}{ccc} (X,\mathcal{X},\mu,T) & \xrightarrow{\phi} & (\hat{X},\hat{\mathcal{X}},\hat{\mu},\hat{T}) \\ \pi\downarrow & & \downarrow\hat{\pi} \\ (Y,\mathcal{Y},\nu,S) & \xrightarrow{\psi} & (\hat{Y},\hat{\mathcal{Y}},\hat{\nu},\hat{S}) \end{array}$$

is commutative (i.e. $\hat{\pi}\circ\phi=\psi\circ\pi$), where ϕ and ψ are measure theoretical isomorphisms.

A classical result in ergodic theory says that for each factor map $\pi:(X,\mathcal{X},\mu,T)\to(Y,\mathcal{Y},\nu,S)$ of ergodic systems there is a factor map $\hat{\pi}:(\hat{X},\hat{T})\to(\hat{Y},\hat{S})$ of Cantor systems such that $\hat{\pi}$ is a model for π (see [3, Theorem 5.15] or [6, Chapter 2]). A natural question is whether one can add additional properties to $\hat{\pi}$.

For example, a consequence of a striking result by Furstenberg and Weiss [4] is that each factor map between two non-periodic ergodic systems has an almost one-to-one model. In this paper, we show that each factor map of two ergodic systems has a model which is either weakly mixing or finite-to-one. A factor map $\pi:(X,T)\to(Y,S)$ between two topological systems is called *(topologically) weakly mixing* if $(R_\pi, T\times T|_{R_\pi})$ is transitive, where $R_\pi=\{(x,y)\in X^2:\pi(x)=\pi(y)\}$. For a natural number N, π is *N-to-1* if Card $\pi^{-1}(\pi(x))=N$ for each $x\in X$.

The notion of weakly mixing extension is important both in ergodic theory and topological dynamics. For example, in ergodic theory, Furstenberg structure theorem [3] says that each ergodic system is a (measurable) weakly mixing extension of a (measurable) distal system. And in topological dynamics, the structure theorem of minimal flows [2, 13, 16] says that the class of minimal flows is the smallest class of flows containing the trivial flow and closed under (a) homomorphisms, (b) inverse limits, and (c) three "building blocks": isometric extensions, proximal extensions and weakly mixing extensions. One can find properties on weakly mixing extensions in [5, 8, 13, 16], and constructions of weakly mixing extensions in [7].

To be precise, here is our main result:

Theorem 1.3. *Let $\pi:(X,\mathcal{X},\mu,T)\to(Y,\mathcal{Y},\nu,T)$ be a factor map with $(X,\mathcal{X},\mu,T)$ ergodic, and let $\mu=\int_{y\in Y}\mu_y d\nu(y)$ be the disintegration of μ over ν. Then one of the following statements holds:*

(1) *if μ_y is non-atomic for ν-a.e. $y\in Y$, then there is a weakly mixing factor map $\hat{\pi}:(\hat{X},\hat{T})\to(\hat{Y},\hat{S})$ which is a model for $\pi:X\to Y$.*
(2) *if μ_y is atomic for ν-a.e. $y\in Y$, then there is a N-to-1 factor map $\hat{\pi}:(\hat{X},\hat{T})\to(\hat{Y},\hat{S})$ which is a model for $\pi:X\to Y$, where $N\in\mathbb{N}$.*

Remark 1.4. In this remark we explain why there are only two possibilities. It is easy to show that $p:X\to\mathbb{R}, x\mapsto\mu_{\pi(x)}(\{x\})$ is measurable and invariant. By the ergodicity, we have p is a constant function. Then we have two cases. The first is when $p=0$ so that μ_y is non-atomic for ν-a.e. $y\in Y$. The second case is when $p>0$ so that μ_y is atomic for ν-a.e. $y\in Y$. For more details, please see the proof of Roklin skew theorem in [6, Page 70].

When Y is trivial, one has the following corollary immediately: Each non-periodic ergodic system has a weakly mixing model. Note that this corollary is a known result. In fact Lehrer [11] showed a much stronger result: Every non-periodic ergodic system has a uniquely ergodic and topologically mixing model.

A more interesting question is as follows, which is a generalization of Weiss's theorem on relatively uniquely ergodic models in [17, 18].

Question 1.5. *In Theorem 1.3, can one require* $(\hat{X}, \hat{T})$ *to be uniquely ergodic?*

We believe that above question has a positive answer, but we could not prove it yet. In Section 4, we will present more related questions.

We organize the paper as follows: in Section 2, we give the basic definitions and facts used in the paper. We prove the main result Theorem 1.3 in Section 3. In the final section, we will discuss unique ergodicity and present some related questions.

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2. Preliminaries

In this section we introduce some notations used in this paper. For more details, please refer to [3, 6].

2.1. A measurable *system* is a quadruple $(X, \mathcal{X}, \mu, T)$ where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is an invertible measure preserving transformation. A measurable system is *ergodic* if all the T-invariant sets have measure either 0 or 1. For an ergodic system, either the space X consists of a finite set of points on which μ is equidistributed, or the measure μ is atom-less. In the first case the system is called *periodic*, and it is called *non-periodic* in the latter.

A *homomorphism* (or called *factor map*) from $(X, \mathcal{X}, \mu, T)$ to a system $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\pi : X_0 \to Y_0$, where X_0 is a T-invariant subset of X and Y_0 is an S-invariant subset of Y, both of full measure, such that $\pi_*\mu = \mu \circ \pi^{-1} = \nu$ and $S \circ \pi(x) = \pi \circ T(x)$ for $x \in X_0$. When we have such a homomorphism we say that the system $(Y, \mathcal{Y}, \nu, S)$ is a *factor* of the system $(X, \mathcal{X}, \mu, T)$. If the factor map $\pi : X_0 \to Y_0$ can be chosen to be bijective, then we say that the systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are *(measure theoretically) isomorphic*. A factor can be characterized (modulo isomorphism) by $\pi^{-1}(\mathcal{Y})$, which is a T-invariant sub-σ-algebra of $\mathcal{X}$.

2.2. A *topological dynamical system* is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. A topological system (X, T) is *transitive* if there exists some point $x \in X$ whose orbit $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ is dense in X. The system is *minimal* if the orbit of any point is dense in X. (X, T) is *topologically weakly mixing* if the product system $(X \times X, T \times T)$ is transitive.

A *factor* of a topological system (X,T) is another topological system (Y,S) such that there exists a continuous and onto map $\phi : X \to Y$ satisfying $S\circ\phi = \phi\circ T$. In this case, (X,T) is called an *extension* of (Y,S). The map ϕ is called a *factor map.*

2.3. Let $(X,\mathcal{X},\mu,T)$ be a measurable system. A *partition* α of X is a family of disjoint measurable subsets of X whose union is X. Let α and β be two partitions of $(X,\mathcal{X},\mu,T)$. One says that α *refines* β, denoted by $\alpha\succ\beta$ or $\beta\prec\alpha$, if each element of β is a union of elements of α.

Let α and β be two partitions. Their *join* is the partition $\alpha\vee\beta = \{A\cap B : A\in\alpha, B\in\beta\}$ and extend this definition naturally to a finite number of partitions. For $m\le n$, define

$$\alpha_m^n = \bigvee_{i=m}^{n} T^{-i}\alpha = T^{-m}\alpha\vee T^{-m+1}\alpha\ldots\vee T^{-n}\alpha,$$

where $T^{-i}\alpha = \{T^{-i}A : A\in\alpha\}$.

2.4. Let $(X,\mathcal{X},\mu,T)$ be an ergodic system and $\alpha = \{A_j\}_{1\le j\le l}$ a finite partition (we usually assume $\mu(A_j)>0$ for all j). We sometimes think of the partition α as a function $\xi_0 : X\to\Sigma = \{1,2,\ldots,l\}$ defined by $\xi_0(x) = j$ for $x\in A_j$. The pair (X,α) is traditionally called a *process*. Let $\Omega = \Omega(l) = \{1,2,\ldots,l\}^{\mathbb{Z}}$ and let S be the shift. One can define a homomorphism ϕ_α from X to Ω, given by $\phi_\alpha(x) = \omega\in\Omega$, where

$$\omega_n = \xi_n(x) = \xi_0(T^n x),\ n\in\mathbb{Z}.$$

We denote the distribution of the stochastic process, $(\phi_\alpha)_*(\mu)$, by $\rho = \rho(X,\alpha)$ and call it the *symbolic representation measure* of (X,α). Let

$$X_\alpha = \mathrm{supp}(\phi_\alpha)_*\mu = \mathrm{supp}\rho.$$

Then we get a homomorphism $\phi_\alpha : (X,\mathcal{X},\mu,T)\to(X_\alpha,\mathcal{X}_\alpha,\rho,S)$. This homomorphism is called the *symbolic representation* of the process (X,α). This will not be a model for $(X,\mathcal{X},\mu,T)$ unless $\bigvee_{i=-\infty}^{\infty} T^{-i}\alpha = \mathcal{X}$ modulo null sets, but in any case this does give a model for a non-trivial factor of X.

2.5. For the set of all finite partitions with the same cardinality, there is a complete metric.

Definition 2.1. Let $(X,\mathcal{X},\mu,T)$ be a system. Let $\alpha = \{A_1,\ldots,A_l\}$ and $\beta = \{B_1,\ldots,B_l\}$ be two l-set partitions ($l\ge 2$), define

$$d_{part}^{\mu}(\alpha,\beta) = \mu(\alpha\Delta\beta) = \frac{1}{2}\sum_{j=1}^{l}\mu(A_j\Delta B_j).$$

Note that $d_{part}^{\mu}(\alpha,\beta)$ will be different when the partitions are indexed in different ways.

3. Proof of Theorem 1.3

In this section, we will prove our main result Theorem 1.3. First we show the difficult case: where the elements of the disintegration are non-atomic. Then we deal with the other case.

3.1. Weakly mixing extensions. In this subsection we prove the first part of Theorem 1.3. That is, we will show the following result.

Proposition 3.1. *Let* $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ *be a factor map with* $(X, \mathcal{X}, \mu, T)$ *ergodic and let* $\mu = \int_{y\in Y} \mu_y d\nu(y)$ *be the disintegration of* μ *over* ν*. If* μ_y *is non-atomic for* ν*-a.e.* $y \in Y$*, then there is a weakly mixing factor map* $\hat{\pi} : \hat{X} \to \hat{Y}$ *which is a model for* $\pi : X \to Y$*.*

$$\begin{array}{ccc} (X, \mathcal{X}, \mu, T) & \xrightarrow{\phi} & (\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T}) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\hat{\pi}}\downarrow \\ (Y, \mathcal{Y}, \nu, S) & \xrightarrow{\psi} & (\hat{Y}, \hat{\mathcal{Y}}, \hat{\nu}, \hat{S}) \end{array}$$

Before going on, we need some preparations. Let $(Y, \mathcal{Y}, \nu, S)$ be a factor of $(X, \mathcal{X}, \mu, T)$. One can identify $L^2(Y, \mathcal{Y}, \nu)$ with the subspace $L^2(X, \pi^{-1}(\mathcal{Y}), \mu)$ of $L^2(X, \mathcal{X}, \mu)$ via $f \mapsto f \circ \pi$. By using this identification it is possible to define the projection of $L^2(X, \mathcal{X}, \mu)$ into $L^2(Y, \mathcal{Y}, \nu)$: $f \mapsto \mathbb{E}(f|\mathcal{Y})$. The *conditional expectation* $\mathbb{E}(f|\mathcal{Y})$ is characterized as the unique $\mathcal{Y}$-measurable function in $L^2(Y, \mathcal{Y}, \nu)$ such that

$$\int_Y g\mathbb{E}(f|\mathcal{Y})d\nu = \int_X g \circ \pi f d\mu \tag{3.1}$$

for all $g \in L^2(Y, \mathcal{Y}, \nu)$.

The *disintegration* of μ over ν is given by a measurable map $y \mapsto \mu_y$ from Y to the space of probability measures on X such that

$$\mathbb{E}(f|\mathcal{Y})(y) = \int_X f d\mu_y \tag{3.2}$$

ν-almost everywhere.

The *self-joining of* $(X, \mathcal{X}, \mu, T)$ *relatively independent over the factor* $(Y, \mathcal{Y}, \nu, S)$ is the system $(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times_Y \mu, T \times T)$, where the measure $\mu \times_Y \mu$ is defined by

$$(\mu \times_Y \mu)(B) = \int_Y \mu_y \times \mu_y(B)d\nu(y), \ \forall B \in \mathcal{X} \times \mathcal{X}. \tag{3.3}$$

This measure is characterized by

$$\int_{X\times X} f_1 \otimes f_2 d\mu \times_Y \mu = \int_Y \mathbb{E}(f_1|\mathcal{Y})\mathbb{E}(f_2|\mathcal{Y})d\nu \tag{3.4}$$

for all $f_1, f_2 \in L^2(X, \mathcal{X}, \mu)$, where $f_1 \otimes f_2(x_1, x_2) = f_1(x_1)f_2(x_2)$.

Definition 3.2. Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map. Let α and β be finite partitions of X and Y respectively with $\alpha \succ \beta$. Let

$$\alpha = \{A_1^1, \dots, A_1^{s_1}, A_2^1, \dots, A_2^{s_2}, \dots, A_k^1, \dots, A_k^{s_k}\},\ \beta = \{B_1, B_2, \dots, B_k\},\ B_i = \bigcup_{j=1}^{s_i} A_i^j.$$

α is called *1-weakly mixing with respect to* β if for any $U, V \in \{A_m^t \times A_n^t : 1 \le t \le k,\ 1 \le m, n \le s_t\}$, there exists some l such that

$$\mu \times_Y \mu(T^l U \cap V) > 0.$$

Let $m \in \mathbb{N}$. A partition α is called *m-weakly mixing with respect to* β if $\alpha_{-(m-1)}^{m-1} = \bigvee_{r=-(m-1)}^{m-1} T^{-r}\alpha$ is 1-weakly mixing with respect to $\beta_{-(m-1)}^{m-1} = \bigvee_{r=-(m-1)}^{m-1} S^{-r}\beta$.

A partition α is called *weakly mixing with respect to* β if for any $m \in \mathbb{N}$, α is m-weakly mixing with respect to β.

Remark 3.3. By a classical abuse of terminology we denote by the same letter the σ-algebra $\mathcal{Y}$ and its inverse image by π. In other words, if $(Y, \mathcal{Y}, \nu, S)$ is a factor of $(X, \mathcal{X}, \mu, T)$, we think of $\mathcal{Y}$ as a sub-σ-algebra of $\mathcal{X}$. Hence in Definition 3.2, $\alpha \succ \beta$ means that $\alpha \succ \pi^{-1}(\beta)$.

By definitions it is easy to verify the following result.

Proposition 3.4. *Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map. Let α be a finite partition of X and β be a finite partition of Y with $\alpha \succ \beta$. Then the symbolic representation X_α of α is an extension of the symbolic representation Y_β of β.*

Let $\hat{\pi}$ be the factor map from X_α to Y_β. Then $\hat{\pi}$ is weakly mixing if α is weakly mixing with respect to β.

The following result is the famous Rohlin Lemma, please refer to [6, 18] for a proof.

Theorem 3.5 (Rohlin Lemma)**.** *Let $(X, \mathcal{X}, \mu, T)$ be a non-periodic ergodic system, N a positive integer and $\epsilon > 0$, then there exists a subset B such that the sets B, $T^{-1}B$, ..., $T^{-(N-1)}B$ are pairwise disjoint and $\mu(\bigcup_{j=0}^{N-1} T^{-j}B) > 1 - \epsilon$.*

To prove Proposition 3.1, we need a generalization of Rohlin Lemma.

Lemma 3.6. *Let $(X, \mathcal{X}, \mu, T)$ be a non-periodic ergodic system. For each $B \in \mathcal{X}$ with $\mu(B) > 0$ and each $n \in \mathbb{N}$, one can find a subset $\hat{B} \subset B$ and $c = c(n) > 0$ such that $\hat{B}, T^{-1}\hat{B}, \dots, T^{-(n-1)}\hat{B}$ are pairwise disjoint, and $\mu(\hat{B}) \ge c\mu(B)$.*

Recall that two sets C, D being disjoint means that $\mu(C \cap D) = 0$.

Proof. Let $N > \frac{4n}{\mu(B)}$, $\epsilon < \frac{\mu(B)}{4}$. By Theorem 3.5, there is a subset C such that C, $T^{-1}C$, ..., $T^{-(N-1)}C$ are pairwise disjoint and $\mu(\bigcup_{j=0}^{N-1} T^{-j}C) > 1 - \epsilon$. Refine the tower $\{C, T^{-1}C, \dots, T^{-(N-1)}C\}$ according to the partition $\{B, X \setminus B\}$. That is, we have a partition $\{C_1, \dots, C_m\}$ of C such that for each $1 \le i \le m$, either $T^{-j}C_i \subset B$ or $T^{-j}C_i \subset X \setminus B$ for all $0 \le j \le N-1$.

Let $i \in \{1, \dots, m\}$. Define $n_{1,i} = \min\{k : 0 \le k \le N-1-n, T^{-k}C_i \subset B\}$ if this set is not empty. Inductively, for $j \ge 2$, if $\{k : n_{j-1,i}+n \le k \le N-1-n, T^{-k}C_i \subset B\}$ is empty, then we are done; if not, then let

$$n_{j,i} = \min\{k : n_{j-1,i} + n \le k \le N-1-n, T^{-k}C_i \subset B\}.$$

Thus we have a finite set $\{n_{1,i}, \dots, n_{s_i,i}\}$ for each $i \in \{1, \dots, m\}$.

Let

$$\hat{B} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{s_i} T^{-n_{j,i}} C_i.$$

By the construction, $\hat{B}, T^{-1}\hat{B}, \dots, T^{-(n-1)}\hat{B}$ are pairwise disjoint. Notice that

$$B \subset (\bigcup_{j=0}^{n-1} T^{-j}\hat{B}) \cup (\bigcup_{j=N-n}^{N-1} T^{-j}C) \cup (X \setminus \bigcup_{j=0}^{N-1} T^{-j}C).$$

Hence

$$\mu(B) < n\mu(\hat{B}) + n\mu(C) + \epsilon < n\mu(\hat{B}) + \frac{n}{N} + \epsilon < n\mu(\hat{B}) + \frac{1}{4}\mu(B) + \frac{1}{4}\mu(B).$$

Then we have $\mu(\hat{B}) > c\mu(B)$, where $c = c(n) = \frac{1}{2n}$. □

Remark 3.7. From the proof of Lemma 3.6, $c = c(n) = \frac{1}{2n}$ is independent of B.

We need the following generalization of Lemma 3.6.

Lemma 3.8. *Let $(X, \mathcal{X}, \mu, T)$ be a non-periodic ergodic system. Let B_1, B_2, ..., B_k be k measurable sets with $\min_{1\le i\le k}\{\mu(B_i)\} = d > 0$ (we do not assume that B_1, B_2, ..., B_k are pairwise disjoint). Then for arbitrary n, there are subsets $\hat{B}_i \subset B_i, \forall i \in \{1, 2, \dots, k\}$, such that*

$$\hat{B}_1, T^{-1}\hat{B}_1, \dots, T^{-(n-1)}\hat{B}_1, \hat{B}_2, T^{-1}\hat{B}_2, \dots, T^{-(n-1)}\hat{B}_2 \dots, \hat{B}_k, T^{-1}\hat{B}_k, \dots, T^{-(n-1)}\hat{B}_k$$

are pairwise disjoint, and $\min_{1\le i\le k}\{\mu(\hat{B}_i)\} \ge cd$, where $c = c(k, n) > 0$ is a constant.

Proof. We make the induction for k. Case $k = 1$ follows from Lemma 3.6. Assume the lemma is right for $k-1$, and now we consider the case of k.

We use inductive assumption on $\{B_1, B_2, \dots, B_{k-1}\}$. Then there are $k-1$ subsets $C_i \subset B_i, 1 \le i \le k-1$, such that

$$C_1, T^{-1}C_1, \dots, T^{-(n-1)}C_1, \dots, C_{k-1}, T^{-1}C_{k-1}, \dots, T^{-(n-1)}C_{k-1}$$

are pairwise disjoint, and $\min_{1\le i\le k-1}\{\mu(C_i)\} \ge c_1 d_{k-1}$, where $c_1 = c_1(k-1, n) > 0$ and $d_{k-1} = \min_{1\le i\le k-1}\{\mu(B_i)\}$. For B_k, by Lemma 3.6 there is a subset $C_k \subset B_k$ and $c_2 = c_2(n) > 0$ such that $C_k, T^{-1}C_k, \dots, T^{-(n-1)}C_k$ are pairwise disjoint and $\mu(C_k) \ge c_2\mu(B_k)$. The problem is that $C_k, T^{-1}C_k, \dots, T^{-(n-1)}C_k$ may intersect the elements from $C_1, T^{-1}C_1, \dots, T^{-(n-1)}C_1, \dots, C_{k-1}, T^{-1}C_{k-1}, \dots, T^{-(n-1)}C_{k-1}$. We should deal with this problem to get what we need.

Let

$$A_{ij} = T^{-i}C_j \cap C_k, A^{ij} = C_j \cap T^{-i}C_k, 0 \le i \le n-1, 1 \le j \le k-1,$$

and

$$A_k = C_k \setminus (\bigcup_{0\le i\le n-1, 1\le j\le k-1} (A_{ij} \cup T^i A^{ij})).$$

Each two elements in $\{A_{ij}, A_k\}_{0\le i\le n-1,1\le j\le k-1}$ are disjoint since Lemma 3.6 and the inductive assumption.

For each A_{ij}, find a subset $D_{ij} \subset A_{ij}$ with $\mu(D_{ij}) = \mu(A_{ij})/2$. Also find a subset $D^{ij} \subset A^{ij}$ with $\mu(D^{ij}) = \mu(A^{ij})/2$. Notice that $A_{0j} = A^{0j}$, so one may assume that $D_{0j} = D^{0j}$. Let

$$\hat{B}_j = C_j \setminus (\bigcup_{i=0}^{n-1} (T^i D_{ij} \cup D^{ij})),\ 1 \le j \le k-1,$$

and let

$$\hat{B}_k = A_k \cup (\bigcup_{0\le i\le n-1, 1\le j\le k-1} (D_{ij} \cup T^i D^{ij})).$$

Let

$$A = \bigcup_{j=1}^{k-1}\bigcup_{i=0}^{n-1} (T^i A_{ij} \cup A^{ij}),\ D = \bigcup_{j=1}^{k-1}\bigcup_{i=0}^{n-1} (T^i D_{ij} \cup D^{ij}),$$

then $\mu(A) = 2\mu(D)$. And $\mu(A \cap C_j) = 2\mu(D \cap C_j)$, for $1 \le j \le k-1$. Notice that

$$C_j = \big(C_j \setminus A\big) \cup \big(D \cap C_j\big) \cup \big((A \setminus D) \cap C_j\big),\ 1 \le j \le k-1,$$

and

$$\hat{B}_j = \big(C_j \setminus A)\big) \cup \big((A \setminus D) \cap C_j\big),\ 1 \le j \le k-1.$$

Then for $d = \min_{1\le j\le k}\{\mu(B_j)\}$,

$$\mu(\hat{B}_j) \ge \mu(C_j)/2 \ge c_1 d_{k-1}/2 \ge c_1 d/2, 1 \le j \le k-1.$$

And

$$\mu(\hat{B}_k) \ge \mu(C_k)/2 \ge c_2 \mu(B_k)/2 \ge c_2 d/2.$$

Hence $\min_{1\le i\le k}\{\mu(\hat{B}_i)\} \ge cd$, where $c = \min\{\frac{1}{2}c_1, \frac{1}{2}c_2\} = c(k, n)$.

Now we prove $\{\hat{B}_1, \hat{B}_2, \dots, \hat{B}_k\}$ satisfies the condition. We only need to consider $T^{-j_1}\hat{B}_i \cap T^{-j_2}\hat{B}_k$, $1 \le i \le k-1$, $0 \le j_1, j_2 \le n-1$.

If $j_1 \le j_2$, $\mu(T^{-j_1}\hat{B}_i \cap T^{-j_2}\hat{B}_k) = \mu(\hat{B}_i \cap T^{-(j_2-j_1)}\hat{B}_k)$. First notice that $\hat{B}_i \cap T^{-(j_2-j_1)}\hat{B}_k \subset A^{i(j_2-j_1)}$. By the construction we have that

$$D^{i(j_2-j_1)} \cap \hat{B}_i = \emptyset, D^{i(j_2-j_1)} \subset T^{-(j_2-j_1)}\hat{B}_k,$$

and

$$A^{i(j_2-j_1)} \setminus D^{i(j_2-j_1)} \subset \hat{B}_i,\ (A^{i(j_2-j_1)} \setminus D^{i(j_2-j_1)}) \cap T^{-(j_2-j_1)}\hat{B}_k = \emptyset.$$

Then

$$\mu(\hat{B}_i \cap T^{-(j_2-j_1)}\hat{B}_k) = \mu((\hat{B}_i \cap A) \cap (T^{-(j_2-j_1)}\hat{B}_k \cap A)) \le \mu((A \setminus D) \cap D) = 0.$$

If $j_1 > j_2$, then $\mu(T^{-j_1}\hat{B}_i \cap T^{-j_2}\hat{B}_k) = \mu(T^{-(j_1-j_2)}\hat{B}_i \cap \hat{B}_k)$. Similarly one can show that $\mu(T^{-(j_1-j_2)}\hat{B}_i \cap \hat{B}_k) = 0$ by the construction of $D_{i(j_1-j_2)}$. □

Remark 3.9. By Remark 3.7 and the proof of Lemma 3.8, one can see that $c(k,n)$ may be chosen as $c(k,n)=\frac{1}{2^{k+1}n}$, which is independent of B_1, B_2, …, B_k.

Finally we need a lemma by Rohlin.

Lemma 3.10. [14, Lemma 3′, No.3 of §4] *Let $\mu=\int_{y\in Y}\mu_y d\nu(y)$ be the disintegration of μ over ν. Suppose that μ_y is non-atomic for ν-a.e. $y\in Y$. If B is a measurable set of X with $\mu_y(B)\geq r>0$ for ν-a.e. $y\in Y$, then for any $0\leq\theta\leq r$ there exists a measurable set B_θ such that $B_\theta\subseteq B$ and $\mu_y(B_\theta)=\theta$ for ν-a.e. $y\in Y$.*

The following proposition is the key to the proof of Proposition 3.1.

Proposition 3.11. *Let $\pi:(X,\mathcal{X},\mu,T)\to(Y,\mathcal{Y},\nu,S)$ be a factor map. Let $\mu=\int_{y\in Y}\mu_y d\nu(y)$ be the disintegration of μ over ν, and assume that μ_y is non-atomic for ν-a.e. $y\in Y$.*

Given a finite partition β of Y and an arbitrary finite partition $\hat{\alpha}$ of X which refines β, for any $\epsilon>0$ there is a partition α of X satisfying

(1) *$\alpha\succ\beta$;*
(2) *$d^{\mu}_{part}(\alpha,\hat{\alpha})<\epsilon$;*
(3) *α is weakly mixing with respect to β.*

Proof. **Step 1:** Since $\hat{\alpha}\succ\beta$, let

$$\hat{\alpha}=\{A_1^1,\ldots,A^{s_{1,1}},A_2^1,\ldots,A_2^{s_{1,2}},\ldots,A_{k_1}^1,\ldots,A_{k_1}^{s_{1,k_1}}\},\ \beta=\{B_1,B_2,\ldots,B_{k_1}\},\ B_i=\bigcup_{j=1}^{s_{1,i}}A_i^j.$$

Let $t_1=\max\{s_{1,1},\ldots,s_{1,k_1}\}$.

Since S is ergodic, for each B_i,B_j $(i,j\in\{1,\ldots,k_1\})$ there is some l_{ij} such that $\nu(B_i\cap S^{l_{ij}}B_j)>0$. Let $\lambda_1=\min_{1\leq i,j\leq k_1}\{\nu(B_i\cap S^{l_{ij}}B_j)\}$.

We need the following fact: for each $B\in\mathcal{Y}$,

$$\mu\times_Y\mu(\pi^{-1}(B)\times\pi^{-1}(B))=\nu(B).$$

It follows from this that

$$\mu\times_Y\mu(\pi^{-1}(B)\times\pi^{-1}(B))=\int_Y\mu_y(\pi^{-1}(B))\mu_y(\pi^{-1}(B))d\nu(y),$$

and $\mu_y(\pi^{-1}(B))=1$ when $y\in B$, $\mu_y(\pi^{-1}(B))=0$ when $y\in Y\setminus B$.

We will use the following claim frequently.

Claim: For each $B\in\mathcal{Y}$, one can find $C,D\subseteq\pi^{-1}(B)$ such that

$$\mu\times_Y\mu(C\times D)\geq\frac{1}{9}\upsilon(B)\quad\text{and}\quad C\cap D=\emptyset. \tag{3.5}$$

Proof of Claim: By Lemma 3.10, there is $C\subset\pi^{-1}(B)$ such that $\frac{1}{2}\geq\mu_y(C)\geq\frac{1}{3}$ for $y\in B$. Then by Lemma 3.10 again, there is $D\subset\pi^{-1}(B)\setminus C$ such that $\mu_y(D)\geq\frac{1}{3}$ for $y\in B$. Hence we have

$$\mu\times_Y\mu(C\times D)=\int_B\mu_y(C)\mu_y(D)d\nu(y)\geq\frac{1}{9}\nu(B).$$

The proof of Claim is completed.

Now for each $i,j\in\{1,\dots,k_1\}$, by Lemma 3.8 there are t_1^4 disjoint subsets $B_{ij}^{\vec{s}}\subset B_i\cap S^{l_{ij}}B_j\in\mathcal{Y}$, where $\vec{s}=(s_1,s_2,s_3,s_4)\in\{1,\dots,t_1\}^4$, such that:

(1) Each two elements in $\{B_{ij}^{\vec{s}}, S^{-l_{ij}}B_{ij}^{\vec{s}'}\}_{i,j,\vec{s},\vec{s}'}$ are disjoint.

(2)
$$0<\nu(B_{ij}^{\vec{s}})<\min\{\frac{\lambda_1}{2t_1^4},\frac{\epsilon}{24k_1^2t_1^4}\},$$
for all $i,j\in\{1,\dots,k_1\}$, $\vec{s}\in\{1,2\dots,t_1\}^4$.

(3) Choose $\epsilon_1>0$ such that $9\epsilon_1<\nu(B_{ij}^{\vec{s}})$ for all $i,j,\vec{s}$.

By Claim there are subsets $C_{ij}^{\vec{s}},D_{ij}^{\vec{s}}\subset\pi^{-1}(B_{ij}^{\vec{s}})$ such that $C_{ij}^{\vec{s}}\cap D_{ij}^{\vec{s}}=\emptyset$ and

$$\mu\times_Y\mu(C_{ij}^{\vec{s}}\times D_{ij}^{\vec{s}})\geq\frac{1}{9}\nu(B_{ij}^{\vec{s}})>\epsilon_1. \tag{3.6}$$

Now we modify the partition $\hat{\alpha}$. Let
$$K_1=\bigcup_{i,j,\vec{s}}\left(C_{ij}^{\vec{s}}\cup D_{ij}^{\vec{s}}\cup T^{-l_{ij}}C_{ij}^{\vec{s}}\cup T^{-l_{ij}}D_{ij}^{\vec{s}}\right)$$

Change A_i^s of $\hat{\alpha}$ to A'^s_i as follows:
$$A'^s_i=(A_i^s\setminus K_1)\cup(\bigcup_{\substack{s_1=s\\1\leq m\leq k_1}}C_{im}^{\vec{s}})\cup(\bigcup_{\substack{s_2=s\\1\leq m\leq k_1}}D_{im}^{\vec{s}})\cup(\bigcup_{\substack{s_3=s\\1\leq n\leq k_1}}T^{-l_{ni}}C_{ni}^{\vec{s}})\cup(\bigcup_{\substack{s_4=s\\1\leq n\leq k_1}}T^{-l_{ni}}D_{ni}^{\vec{s}})$$

Then we have a new partition $\hat{\alpha}_1=\{A'^1_1,\dots,A'^{s_{1,1}}_1,A'^1_2,\dots,A'^{s_{1,2}}_2,\dots,A'^1_{k_1},\dots,A'^{s_{1,k_1}}_{k_1}\}$, which also refines β. We have that $d^\mu_{part}(\hat{\alpha},\hat{\alpha}_1)<\epsilon/6$, since
$$\mu(K_1)<4k_1^2t_1^4\max_{i,j,\vec{s}}\nu(B_{ij}^{\vec{s}})<\frac{\epsilon}{6}.$$

By the construction, for each $\vec{s}=(s_1,s_2,s_3,s_4)$ and each $i,j\in\{1,\dots,k\}$ one has that
$$A'^{s_1}_i\times A'^{s_2}_i\cap(T\times T)^{l_{ij}}(A'^{s_3}_j\times A'^{s_4}_j)\supset C_{ij}^{\vec{s}}\times D_{ij}^{\vec{s}}.$$

And by (3.6)
$$\mu\times_Y\mu(A'^{s_1}_i\times A'^{s_2}_i\cap(T\times T)^{l_{ij}}(A'^{s_3}_j\times A'^{s_4}_j))\geq\mu\times_Y\mu(C_{ij}^{\vec{s}}\times D_{ij}^{\vec{s}})>\epsilon_1>0.$$

In particular, $\hat{\alpha}_1$ is 1-weakly mixing with respect to β.

By induction, we have a sequence of partitions $\hat{\alpha}_0=\hat{\alpha},\hat{\alpha}_1,\dots,\hat{\alpha}_{n-1}$ of X, and a sequence of positive numbers $\epsilon_0=\epsilon,\epsilon_1,\dots,\epsilon_{n-1}$ such that for each $i\in\{1,\dots,n-1\}$

(1_i) $\hat{\alpha}_i\succ\beta$, $0<\epsilon_i<\dfrac{\epsilon_{i-1}}{100}$;

(2_i) $d^\mu_{part}(\hat{\alpha}_{i-1},\hat{\alpha}_i)<\dfrac{\epsilon_{i-1}}{6}$;

(3_i) For each $A_1, A_2, A_3, A_4 \in (\hat{\alpha}_i)_{-(i-1)}^{i-1} = \bigvee_{t=-(i-1)}^{i-1} T^{-t}\hat{\alpha}_i$ with A_1, A_2 and A_3, A_4 being subsets of the same element of $\beta_{-(i-1)}^{i-1} = \bigvee_{t=-(i-1)}^{i-1} S^{-t}\beta$ respectively, there is some l such that

$$\mu \times_Y \mu\big(A_1 \times A_2 \cap (T^l \times T^l)(A_3 \times A_4)\big) > \epsilon_i.$$

In particular, $\hat{\alpha}_i$ is i-weakly mixing with respect to β.

Step n: Now let us construct $\hat{\alpha}_n$. Since $\hat{\alpha}_{n-1} \succ \beta$, we have $(\hat{\alpha}_{n-1})_{-(n-1)}^{n-1} \succ \beta_{-(n-1)}^{n-1}$. Let

$\beta_{-(n-1)}^{n-1} = \{F_1, F_2, \dots, F_{k_n}\}$, $(\hat{\alpha}_{n-1})_{-(n-1)}^{n-1} = \{E_1^1, \dots, E_1^{s_{n,1}}, E_2^1, \dots, E_2^{s_{n,2}}, \dots, E_{k_n}^1, \dots, E_{k_n}^{s_{n,k_n}}\}$ such that $F_i = \bigcup_{j=1}^{s_{n,i}} E_i^j, 1 \le i \le k_n$. Let $t_n = \max\{s_{n,1}, \dots, s_{n,k_n}\}$.

Since Y is ergodic, for each F_i, F_j $(i, j \in \{1, \dots, k_n\})$ there is some h_{ij} such that $\nu(F_i \cap S^{h_{ij}} F_j) > 0$. Let $\lambda_n = \min_{1 \le i,j \le k_n}\{\nu(F_i \cap S^{h_{ij}} F_j)\}$.

Now for each $i, j \in \{1, \dots, k_n\}$, according to Lemma 3.8, there are t_n^4 disjoint subsets $F_{ij}^{\vec{s}} \subset F_i \cap S^{h_{ij}} F_j \in \mathcal{Y}$, where $\vec{s} = (s_1, s_2, s_3, s_4) \in \{1, 2 \dots, t_n\}^4$, such that:

(1) Each two elements in $\{S^{-t}F_{ij}^{\vec{s}}, S^{-h_{ij}-t}F_{ij}^{\vec{s}'} : -(n-1) \le t \le n-1, i, j, \vec{s}\}$ are disjoint.

(2)
$$0 < \nu(F_{ij}^{\vec{s}}) < \min\{\frac{\lambda_n}{2c(k_n, n)t_n^4}, \frac{\epsilon_{n-1}}{24(2n-1)k_n^2 t_n^4}\} < \frac{\epsilon_{n-1}}{24},$$
for all $i, j \in \{1, \dots, k_n\}$, $\vec{s} \in \{1, 2 \dots, t_n\}^4$, where $c(k_n, n)$ is a constant as in Lemma 3.8.

(3) Choose $\epsilon_n > 0$ such that $9\epsilon_n < \nu(F_{ij}^{\vec{s}})$ for all $i, j, \vec{s}$.

By Claim for all $i, j \in \{1, \dots, k_n\}$ and $\vec{s} \in \{1, \dots, t_n\}^4$ there are $G_{ij}^{\vec{s}}, H_{ij}^{\vec{s}} \subset \pi^{-1}(F_{ij}^{\vec{s}})$ such that $G_{ij}^{\vec{s}} \cap H_{ij}^{\vec{s}} = \emptyset$ and

$$\mu \times_Y \mu(G_{ij}^{\vec{s}} \times H_{ij}^{\vec{s}}) \ge \frac{1}{9}\nu(F_{ij}^{\vec{s}}) > \epsilon_n. \tag{3.7}$$

Now we modify the partition $\hat{\alpha}_{n-1} = \{U_1, U_2, \dots, U_p\}$ to $\hat{\alpha}_n = \{U_1', U_2', \dots, U_p'\}$, where $p = \sum_{i=1}^k s_{1,i}$ since all $\hat{\alpha}_i$ have the same cardinality. Let

$$K_n = \bigcup_{i,j,-(n-1)\le t\le n-1,\vec{s}} \big(T^{-t}(G_{ij}^{\vec{s}} \cup H_{ij}^{\vec{s}}) \cup T^{-h_{ij}-t}(G_{ij}^{\vec{s}} \cup H_{ij}^{\vec{s}})\big)$$

To get $\hat{\alpha}_n$, we cut K_n away from every element of $\hat{\alpha}_{n-1}$ to get a set $\{U_1 \setminus K_n, U_2 \setminus K_n, \dots, U_p \setminus K_n\}$. Then we add some suitable elements from

$$\big\{T^{-t}G_{ij}^{\vec{s}}, T^{-t}H_{ij}^{\vec{s}}, T^{-h_{ij}-t}G_{ij}^{\vec{s}}, T^{-h_{ij}-t}H_{ij}^{\vec{s}}\big\}_{i,j,-(n-1)\le t\le n-1,\vec{s}}$$

to each $U_j \setminus K_n$ such that the resulting sets meet our needs.

For each $i \in \{1, \dots, k_n\}$ and $s \in \{1, \dots, s_i^n\}$, $E_i^s \in (\hat{\alpha}_{n-1})_{-(n-1)}^{n-1}$ has a $\{1, 2, \dots, p\}^{2n-1}$ name with respect to $\hat{\alpha}_{n-1}$. Denote it by $(d_{-(n-1)}, d_{-(n-2)}, \dots, d_{n-1})$. That is

$$E_i^s = T^{n-1}U_{d_{-(n-1)}} \cap T^{n-2}U_{d_{-(n-2)}} \cap \ldots \cap T^{-(n-1)}U_{d_{n-1}}.$$

Then for each $-(n-1) \le u \le n-1$, if $d_u = j$, we add the following set to $U_j \setminus K_n$:

$$\big(\bigcup_{\substack{s_1=s\\1\le r\le k_n}} T^u G_{ir}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_2=s\\1\le r\le k_n}} T^u H_{ir}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_3=s\\1\le t\le k_n}} T^{u-h_{ti}} G_{ti}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_4=s\\1\le t\le k_n}} T^{u-h_{ti}} H_{ti}^{\vec{s}}\big)$$

That is, for the resulting partition $\hat{\alpha}_n = \{U'_1, U'_2, \dots, U'_p\}$, we have

$$\begin{aligned}&(U_j \setminus K_n) \cup \big(\bigcup_{\substack{s_1=s\\1\le r\le k_n}} T^u G_{ir}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_2=s\\1\le r\le k_n}} T^u H_{ir}^{\vec{s}}\big)\\ &\cup \big(\bigcup_{\substack{s_3=s\\1\le t\le k_n}} T^{u-h_{ti}} G_{ti}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_4=s\\1\le t\le k_n}} T^{u-h_{ti}} H_{ti}^{\vec{s}}\big) \subset U'_j.\end{aligned} \tag{3.8}$$

Now we show the partition $\hat{\alpha}_n = \{U'_1, U'_2, \dots, U'_p\}$ is what we need. First notice that $d^{\mu}_{part}(\hat{\alpha}_{n-1}, \hat{\alpha}_n) < \frac{\epsilon_{n-1}}{6}$, since

$$\mu(K_n) < 4(2n-1)k_n^2 t_n^4 \max_{i,j,\vec{s}} \nu(F_{ij}^{\vec{s}}) < \frac{\epsilon_{n-1}}{6}.$$

Let

$$(\hat{\alpha}_n)_{-(n-1)}^{n-1} = \{E'^1_1, \dots, E'^{s_{n,1}}_1, E'^1_2, \dots, E'^{s_{n,2}}_2, \dots, E'^1_{k_n}, \dots, E'^{s_{n,k_n}}_{k_n}\},$$

where E'^s_i has the same $\{1, \dots, p\}^n$ name with E^s_i for each $i \in \{1, \dots, k_n\}$ and $s \in \{1, 2, \dots, s_{n,i}\}$. Denote the name of E'^s_i by $(d_{-(n-1)}, d_{-(n-2)}, \dots, d_{n-1})$. Then by (3.8), we have

$$\begin{aligned}&\big(\bigcup_{\substack{s_1=s\\1\le r\le k_n}} G_{ir}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_2=s\\1\le r\le k_n}} H_{ir}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_3=s\\1\le t\le k_n}} T^{-h_{ti}} G_{ti}^{\vec{s}}\big) \cup \big(\bigcup_{\substack{s_4=s\\1\le t\le k_n}} T^{-h_{ti}} H_{ti}^{\vec{s}}\big)\\ &\subset T^{n-1} U'_{d_{-(n-1)}} \cap T^{n-2} U'_{d_{-(n-2)}} \cap \dots \cap T^{-(n-1)} U'_{d_{n-1}}\\ &= E'^s_i.\end{aligned}$$

Thus for each $i, j \in \{1, \dots, k_n\}$, $s_1, s_2 \in \{1, \dots, s_{n,i}\}$ and $s_3, s_4 \in \{1, \dots, s_{n,j}\}$, we have

$$E'^{s_1}_i \times E'^{s_2}_i \cap (T \times T)^{h_{ij}} (E'^{s_3}_j \times E'^{s_4}_j) \supset G_{ij}^{\vec{s}} \times H_{ij}^{\vec{s}},$$

where $\vec{s} = (s_1, s_2, s_3, s_4)$. Hence

$$\begin{aligned}&\mu \times_Y \mu(E'^{s_1}_i \times E'^{s_2}_i \cap (T \times T)^{h_{ij}} (E'^{s_3}_j \times E'^{s_4}_j))\\ \ge\ &\mu \times_Y \mu(G_{ij}^{\vec{s}} \times H_{ij}^{\vec{s}}) > \epsilon_n > 0.\end{aligned}$$

That is, $\hat{\alpha}_n$ satisfies conditions (1_n), (2_n) and (3_n), and we finish our induction.

To sum up, we have a sequence of partitions $\{\hat{\alpha}_n\}_{n=1}^{\infty}$ of X, and a sequence of positive numbers $\{\epsilon_n\}_{n=1}^{\infty}$ satisfying conditions (1_n), (2_n) and (3_n) for each n. Notice that by (1_n), $\sum_n \epsilon_n < \epsilon/2$.

By (2_n), $d^{\mu}_{part}(\hat{\alpha}_{n-1}, \hat{\alpha}_n) < \epsilon_{n-1}/6$, and hence $\{\hat{\alpha}_n\}$ is a Cauchy sequence since $\sum_n \epsilon_n < \epsilon/2$. Since d^{μ}_{part} is a complete metric, let $\alpha = \lim_n \hat{\alpha}_n$. Then $d^{\mu}_{part}(\alpha, \hat{\alpha}) < \epsilon$ and by (1_n) α refines β. Now we show that α is weakly mixing with respect to β.

Let $n \in \mathbb{N}$ and let $A_1, A_2, A_3, A_4 \in \alpha_{-(n-1)}^{n-1} = \bigvee_{t=-(n-1)}^{n-1} T^{-t}\alpha$ with A_1, A_2 and A_3, A_4 being subsets of the same element of $\beta_{-(n-1)}^{n-1}$ respectively. There is some $m > n$ such that $d_{part}^{\mu}(\hat{\alpha}_m, \alpha) < \epsilon_{n+1}$. Let $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4 \in (\hat{\alpha}_m)_{-(n-1)}^{n-1}$ be the corresponding sets of A_1, A_2, A_3, A_4 in $\alpha_{-(n-1)}^{n-1}$.

Notice that for any set $K, A, B \in \mathcal{X}$ with $\mu(K) < \epsilon/6$, we have

$$\mu \times_Y \mu((A \setminus K) \times (B \setminus K)) > \mu \times_Y \mu(A \times B) - 3\mu(K) = \mu \times_Y \mu(A \times B) - \frac{\epsilon}{2}.$$

By the construction, we have

$$\mu \times_Y \mu(\hat{A}_1 \times \hat{A}_2 \cap (T^l \times T^l)(\hat{A}_3 \times \hat{A}_4)) > \epsilon_n - \frac{\epsilon_n}{2} - \ldots - \frac{\epsilon_{m-1}}{2}.$$

Thus

$$\begin{aligned} & \mu \times_Y \mu(A_1 \times A_2 \cap (T^l \times T^l)(A_3 \times A_4)) \\ > \;& \mu \times_Y \mu(\hat{A}_1 \times \hat{A}_2 \cap (T^l \times T^l)(\hat{A}_3 \times \hat{A}_4)) - 3\epsilon_{n+1} \\ > \;& \epsilon_n - \frac{\epsilon_n}{2} - \ldots - \frac{\epsilon_{m-1}}{2} - 3\epsilon_{n+1} > 0, \end{aligned}$$

since $\epsilon_{j+1} < \frac{\epsilon_j}{100}$ for arbitrary j. That means α is weakly mixing with respect to β. The proof is completed. □

Now it is time to finish the proof of Proposition 3.1. To finish the proof, we need use the proof of Proposition 3.11 to get an increasing sequence of required partitions γ_n such that the inverse limit of the corresponding symbolic representations is what we need. This part of the proof is standard (see, for example, [11, 15]).

Proof of Proposition 3.1. Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map with $(X, \mathcal{X}, \mu, T)$ ergodic. Let $\{\alpha_n\}, \{\beta_n\}$ be two increasing sequences of partitions such that $\alpha_n \succ \beta_n$ for all $n \in \mathbb{N}$ and $\sigma(\beta_n) \nearrow \mathcal{Y}$, $\sigma(\alpha_n) \nearrow \mathcal{X}$, where $\sigma(\gamma)$ is the σ-algebra generated by the set γ.

For α_1, we adjust α_1 as in Step 1 of the proof of Proposition 3.11 to get a new partition γ_1^1. We replace α_2 by $\alpha_2 \bigvee \gamma_1^1$, and thus we have $\gamma_1^1 \prec \alpha_2$. We adjust $\bigvee_{i=-1}^{1} T^{-i}\alpha_2$ as in Step 2 of the proof of Proposition 3.11 to get a new partition γ_2^2. Inductively, we replace α_n by $\alpha_n \bigvee \gamma_{n-1}^{n-1}$, and we adjust $\bigvee_{i=-(n-1)}^{n-1} T^{-i}\alpha_n$ as in Step n of the proof of Proposition 3.11 to get a new partition γ_n^n.

Now we define inductively γ_k^n for $k < n$. When $n = 1$ there is nothing to say. We assume that we have done for $n-1$ ($n \geq 2$), i.e. the partitions $\gamma_1^{n-1} \prec \ldots \prec \gamma_{n-1}^{n-1}$ are given.

Let $k < n$, $\alpha_n = \{A_1, \ldots, A_a\}$, $\gamma_n^n = \{A_1', \ldots, A_a'\}$ and $\gamma_k^{n-1} = \{B_1, \ldots, B_b\}$. Since $\alpha_n \succ \gamma_k^{n-1}$, there is a function

$$\phi : \{1, \ldots, b\} \to 2^{\{1,\ldots,a\}} \backslash \emptyset$$

such that $A_x \subset B_y$ means $x \in \phi(y)$. And let

$$\gamma_k^n = \{B_1', \ldots, B_a'\}, B_s' = \bigcup_{t \in \phi(s)} A_t'.$$

Hence we have partitions $\{\gamma_k^n\}_{n\in\mathbb{N},1\le k\le n}$ via $\{\gamma_n^n\}_n$. Let X_k^n denote the corresponding symbolic system of γ_k^n. The array shows the induction.

$$\begin{array}{llll} X_1^1 & & & \\ X_1^2 & X_2^2 & & \\ X_1^3 & X_2^3 & X_3^3 & \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be as appeared in the proof of Proposition 3.11. Then for $k > n$ we have $d(\gamma_n^k, \gamma_n^{k-1}) < \frac{1}{6}\epsilon_k$. That means for each n, $\{\gamma_n^k\}_{k\ge n}$ is a Cauchy sequence. So there is a partition γ_n such that $\gamma_n^k \to \gamma_n$, as $k \to \infty$. Now we have an increasing sequence $\{\gamma_n\}_n$ satisfying for each n

(a) $\gamma_n \succ \beta_n$.
(b) $d^{\mu}_{part}(\gamma_n, \alpha_n) < \frac{1}{6}\sum_{k=n}^{\infty}\epsilon_k$.
(c) γ_n is n-weakly mixing with respect to β_n.

Let $X_n = X_{\gamma_n}$ be the symbolic representation of γ_n. Since $\sigma(\alpha_n) \nearrow \mathcal{X}$ and (b), we have $\sigma(\gamma_n) \nearrow \mathcal{X}$. Denote $\hat{\pi} : (\hat{X}, \hat{T}) \to (\hat{Y}, \hat{S})$ the inverse limit of $\pi_n : (X_n, T_n) \to (Y_n, S_n)$, where $Y_n = Y_{\beta_n}$.

$$\begin{array}{ccccccccc} (\hat{X},\hat{T}) & \longrightarrow & \ldots & \longrightarrow & (X_3,T_3) & \longrightarrow & (X_2,T_2) & \longrightarrow & (X_1,T_1) \\ \downarrow{\scriptstyle \hat{\pi}} & & \downarrow & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ (\hat{Y},\hat{S}) & \longrightarrow & \ldots & \longrightarrow & (Y_3,S_3) & \longrightarrow & (Y_2,S_2) & \longrightarrow & (Y_1,S_1) \end{array}$$

Then $\hat{\pi} : (\hat{X}, \hat{T}) \to (\hat{Y}, \hat{S})$ is a model for $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$, and by (c) $\hat{\pi}$ is weakly mixing. The proof is completed. □

3.2. **Finite to one extensions.** In this subsection, we show the second part of Theorem 1.3.

Proposition 3.12. *Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map with $(X, \mathcal{X}, \mu, T)$ ergodic and let $\mu = \int_{y\in Y}\mu_y d\nu(y)$ be the disintegration of μ over ν. If μ_y is atomic for ν-a.e. $y \in Y$, then there is a N-to-1 extension factor map $\hat{\pi} : (\hat{X}, \hat{T}) \to (\hat{Y}, \hat{S})$ which is a model for $\pi : X \to Y$.*

Proof. By Rohlin's skew-product theorem, we may assume that

$$(X, \mathcal{X}, \mu, T) = Y \times_{\omega} (U, \rho) = (Y \times U, \mathcal{Y} \times \mathcal{U}, \nu \times \rho, T_{\omega})$$

and $\pi : X \to Y$ is the projection. Recall that $\omega : Y \to \mathrm{Aut}(U, \rho)$ is a measurable cocycle, and

$$T_{\omega}(y, u) = (Sy, \omega(y)u).$$

By our assumption in the proposition, we have that $U = \{1, 2, \ldots, N\}$ is a finite set, $\mathcal{U} = 2^U$, and $\rho(\{i\}) = \frac{1}{N}$ for all $i \in U$.

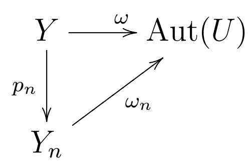
Since U is a finite set, $\mathrm{Aut}(U,\rho)$ is also a finite set. Assume $\mathrm{Aut}(U,\rho) = \{\eta_1, \eta_2, \dots \eta_k\}$, and let

$$E_i = \omega^{-1}(\{\eta_i\}) = \{y \in Y : \omega(y) = \eta_i(y)\},\ 1 \le i \le k.$$

Then E_i is measurable for each $1 \le i \le k$ and $\xi = \{E_1, E_2, \dots, E_k\}$ is a partition of Y.

Let $\{\gamma_n\}_n$ be an increasing sequence of finite partitions with $\sigma(\gamma_n) \nearrow \mathcal{Y}$. Let $\beta_n = \gamma_n \vee \xi$. Then we also have that $\sigma(\beta_n) \nearrow \mathcal{Y}$. Let $Y_n = Y_{\beta_n}$ be the symbolic representation of β_n and $\nu_n = \rho(Y, \beta_n)$. Then the inverse limit $\hat{Y} = \varprojlim Y_n$ is a model for $(Y, \mathcal{Y}, \nu, S)$. Let $p_n : Y \to Y_n$ be the corresponding factor map. Now we deduce a cocycle $\omega_n : Y_n \to \mathrm{Aut}(U)$ from ω. Since for each $y \in Y_n$, $p_n^{-1}(y) \subset E_j$ for some $j \in \{1, \dots, k\}$, it is reasonable to define $\omega_n : Y_n \to \mathrm{Aut}(U)$ as follows:

$$\omega_n(y) = \omega(z),\ z \in p_n^{-1}(y).$$



Let $X_n = Y_n \times U$, $\mathcal{X}_n = \mathcal{Y}_n \times \mathcal{U}$, $\mu_n = \nu_n \times \rho$, and $T_{\omega_n} : X_n \to X_n$

$$T_{\omega_n}(y, u) = (Sy, \omega_n(y)u).$$

Since $\hat{Y} = \varprojlim Y_n$ is a model for $(Y, \mathcal{Y}, \nu, S)$, $\hat{X} = \varprojlim X_n = \varprojlim Y_n \times U$ is a model for $(X, \mathcal{X}, \mu, T) = (Y \times U, \mathcal{Y} \times \mathcal{U}, \nu \times \rho, T_\omega)$, and the projection $\hat{\pi} : \hat{X} \to \hat{Y}$ is a model for $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$. Since for each $y \in \hat{Y}$, $\hat{\pi}^{-1}(y) = \{y\} \times U$, $\hat{\pi}$ is a N-to-1 extension. The proof is completed. □

4. Unique ergodicity and Questions

Theory on uniquely ergodic models plays a very important role in ergodic theory and topological dynamics. In some sense it makes ergodic theory embed into topological dynamics. Please refer to [6, 8, 17, 18] for more information on this topic.

First recall the definition of unique ergodicity.

Definition 4.1. A topological system (X, T) is called *uniquely ergodic* if there is a unique T-invariant probability measure on X. It is called *strictly ergodic* if it is uniquely ergodic and minimal.

The famous Jewett-Krieger Theorem [9, 10] says that every ergodic system has a uniquely ergodic model. B. Weiss generalized this theorem to the relative case.

Theorem 4.2 (B. Weiss)**.** [17] *If $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ is a factor map with $(X, \mathcal{X}, \mu, T)$ ergodic and $(\hat{Y}, \hat{\mathcal{Y}}, \hat{\nu}, \hat{S})$ is a uniquely ergodic model for $(Y, \mathcal{Y}, \nu, S)$,* *then there is a uniquely ergodic model* $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, \hat{T})$ *for* $(X, \mathcal{X}, \mu, T)$ *and a factor map* $\hat{\pi} : \hat{X} \to \hat{Y}$ *which is a model for* $\pi : X \to Y$.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \hat{X} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\hat{\pi}} \\ Y & \xrightarrow{\psi} & \hat{Y} \end{array}$$

It is natural to ask whether one can add some additional properties on $\hat{\pi}$. For example, in the absolute case, E. Lehrer [11] showed that one can strengthen Jewett-Krieger Theorem as follows: Every ergodic system has a uniquely ergodic and topologically mixing model. And in [8], lots of these kinds of results are discussed.

As to the relative case, there are not too many results. As mentioned in the Section 1, a consequence of Furstenberg-Weiss theorem [4] says that in Weiss's theorem (Theorem 4.2) one may require $\hat{\pi}$ being almost one-to-one when Y is non-periodic. Recently, Béguin, Crovisier and Le Roux showed that one can say more about $\hat{\pi}$ in Weiss's theorem. For example, openness can be achieved:

Theorem 4.3 (Béguin-Crovisier-Le Roux)**.** [1, Theorem A.2] *If in Weiss's Theorem* $(\hat{Y}, \hat{\mathcal{Y}}, \hat{\nu}, \hat{T})$ *is a Cantor system, then* $\hat{\pi}$ *can be open in addition.*

As mentioned in Question 1.5, it seems reasonable to add unique ergodicity in Theorem 1.3. We also think the following properties are reasonable to be added in Weiss's theorem.

Question 4.4. *Is it possible to require* $\hat{\pi}$ *satisfying one of following conditions in Weiss's theorem when* Y *is non-periodic and* π *is not measure finite-to-one:*

(i) *proximal and open;*
(ii) *weakly mixing and RIC.*

Remark 4.5. (1) Recall that for a factor map $\pi : (X, T) \to (Y, S)$, π is *proximal* if for all $x_1, x_2 \in R_\pi = \{(x, y) \in X^2 : \pi(x) = \pi(y)\}$, x_1, x_2 are proximal, i.e. $\liminf_n d(T^n x_1, T^n x_2) = 0$; π is *distal* if for all $x_1, x_2 \in R_\pi$, x_1, x_2 are not proximal; and π is *RIC* (relatively incontractible) if it is open and for every $n > 1$ the minimal points are dense in the relation

$$R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall 1 \le i \le j \le n\}.$$

(2) In Question 4.4, the condition in (i) implies weak mixing, i.e. any open proximal factor map of minimal systems is weakly mixing [5]. The difference between (i) and (ii) is that in (i) the diagonal Δ_X is the only minimal set of $(R_\pi, T \times T)$, but in (ii) minimal points are dense in $(R_\pi, T \times T)$.

Note that not any dynamical properties can be added in the uniquely ergodic models. For example, Lindernstrauss showed that every ergodic measure distal system $(X, \mathcal{X}, \mu, T)$ has a minimal topologically distal model [12]. This topological model need not, in general, be uniquely ergodic. In other words there are measure distal systems for which no uniquely ergodic topologically distal model exists [12]. For more information please refer to [8].

References

[1] F. Béguin, S. Crovisier and F. Le Roux, *Realisation of measured dynamics as uniquely ergodic minimal homeomorphisms on manifolds*, Math. Z. 270 (2012), 59–102.

[2] R. Ellis, S. Glasner and L.P. Shapiro, *Proximal-isometric flows*, Advances in Math., 17(1975), no. 3, 213–260.

[3] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[4] H. Furstenberg and B. Weiss, *On almost* 1-1 *extensions*, Israel J. Math. 65 (1989), no. 3, 311–322.

[5] E. Glasner. *Topological weak mixing and quasi-Bohr systems.* Probability in mathematics. Israel J. Math. 148 (2005), 277–304.

[6] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[7] S. Glasner and B. Weiss, *On the construction of minimal skew products*, Israel J. Math. 34 (1979), no. 4, 321–336.

[8] E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics*, Handbook of dynamical systems. Vol. 1B, 597–648, Elsevier B. V., Amsterdam, 2006.

[9] R.I. Jewett, *The prevalence of uniquely ergodic systems*, J. Math. Mech. 19 (1969/1970), 717–729.

[10] W. Krieger, *On unique ergodicity*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 327–346, Univ. California Press, Berkeley, Calif., 1972.

[11] E. Lehrer, *Topological mixing and uniquely ergodic systems*, Israel J. Math. 57 (1987), no. 2, 239–255.

[12] E. Lindenstrauss, *Measurable distal and topological distal systems*, Ergodic Theory Dynamical Systems 19 (1999), 1063–1076.

[13] D. C. McMahon, *Weak mixing and a note on a structure theorem for minimal transformation groups*, Illinois J. Math. 20 (1976), no. 2, 186–197.

[14] V.A. Rohlin, *On the fundament ideas of measure theory*, Matematicheskii Sbornik (N.S.) 25(67) 1 (1949), 107–150; Engl. transl., American Mathematical Society Translations, Series1 10 (1962), 1–54.

[15] A. Rosenthal, *On strictly ergodic models for commuting ergodic transformations.* Ann. Inst. H. Poincar Probab. Statist. 25 (1989), no. 1, 73–92.

[16] W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. 83, (1977), 775–830.

[17] B. Weiss, *Strictly ergodic models for dynamical systems*, Bull. Amer. Math. Soc. (N.S.) 13 (1985), no. 2, 143–146.

[18] B. Weiss, *Single Orbit Dynamics*, AMS Regional Conference Series in Mathematics, N, 95, 2000.

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