STRUCTURE OF BOUNDED TOPOLOGICAL-SEQUENCE-ENTROPY MINIMAL SYSTEMS

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Abstract. In this article we prove that a minimal topological dynamical system (X,T) with bounded topological sequence entropy has the following structure:

$$\begin{array}{ccc} X & \xleftarrow{\sigma'} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X_{eq} & \xleftarrow{\tau'} & Y' \end{array}$$

where π is the maximal equicontinuous factor of (X,T), σ' and τ' are proximal extensions and π' is a finite to one equicontinuous extension. In order to prove this result we consider sequence entropy tuples and we give a complete relation of them with regionally proximal tuples.

1. Introduction

As in ergodic theory one of the main tools to study the dynamical behavior of a topological dynamical system (i.e. a homeomorphism $T : X \to X$ where X is a compact metric space) is to understand its fundamental factors and extensions. Most of them constructed from invariant relations defined between pairs of points in the system. Among such factors the maximal equicontinuous one has played a crucial role to understand continuous eigenvalues and also the so called complexity relations among points of the system [BHM]. The structure of the maximal equicontinuous factor has attracted a lot of attention. One motivation is to classify zero entropy systems.

Sequence entropy for a measure was introduced as an isomorphism invariant by Kushnirenko [Ku], who used it to distinguish between transformations with the same entropy and spectral invariant. It was also shown that an invertible measure preserving transformation has discrete spectrum if and only if for any sequence the sequence entropy of the system is zero [Ku]. Let $(X,\mathcal{X},\mu,T)$ be an ergodic system and $(Z,\mathcal{Z},m,S)$ be its Kronecker factor. Then by Rohlin Theorem, T may be regarded as a skew product on $Z \times K$, where either K consists of k atoms of measure $1/k$ or it is continuous. It was shown that the supremum of all sequence entropies of T is $\log k$ in the first case and infinite in the second [Hu]. In an analogous way Goodman introduced the topological sequence entropy [Go].

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In this article we study the structure of the maximal equicontinuous factor of a minimal topological dynamical system of bounded topological sequence entropy. That is, there is a positive real number H such that for each increasing sequence of positive integer numbers, the topological entropy of the system along this sequence is bounded by H. This class of systems extends the notion of null system introduced in [HLSY] where $H = 0$. We prove that this factor can be lifted by proximal extensions to a finite to one equicontinuous extension. A natural question is whether the proximal extensions can be replaced by almost one to one extensions. We could not solve this question in this article. It is interesting to remark that weakly mixing systems have unbounded topological sequence entropy and in particular their maximal equicontinuous factor is trivial.

The classical way to construct the maximal equicontinuous factor of a system is from the regionally proximal relation (see [Au, G1, Vr]). From sequence entropy one defines topological sequence entropy pairs, which allow to construct the maximal null factor [HLSY]. To understand the structure of the maximal equicontinuous factor we consider in Section 3 the notions of n-sequence entropy tuples and n-regionally proximal tuples, and we give a complete description of the inclusions between them. Sequence entropy tuples for a measure were studied in [HMY].

The article is organized as follows. In Section 2 we give some basic background in topological dynamics. Then in Section 3 we develop the notion of n-sequence entropy tuples. The structure theorem is proved in Section 4. Finally in Section 5 we deduce some sufficient conditions to have unbounded topological sequence entropy.

2. Preliminaries

In the article, integers, nonnegative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ respectively. In the following subsections we give the basic background in topological dynamics necessary for the article. More details can be found in [Au, Br, G1, Vr].

2.1. **Topological dynamical systems and factors.** By a *topological dynamical system* (TDS for short) we mean a pair (X, T) where X is a compact metric space (with metric d) and $T : X \to X$ is a homeomorphism. The *orbit of* $x \in X$ is given by $Orb(x, T) = \{T^n x : n \in \mathbb{Z}\}$. For $n \geq 2$ one writes $(X^n, T^{(n)})$ for the n-fold product system $(X \times \cdots \times X, T \times \cdots \times T)$. The diagonal of X^n is $\Delta_n(X) = \{(x, \ldots, x) \in X^n : x \in X\}$ and $\Delta^{(n)}(X) = \{(x_1, \ldots, x_n) \in X^n : \text{ for some } i \neq j\ , x_i = x_j\}$. When $n = 2$ one writes $\Delta_2(X) = \Delta(X)$.

A TDS (X, T) is *transitive* if for any two nonempty open sets U and V there is $n \in \mathbb{Z}$ such that $U \cap T^{-n}V \neq \emptyset$. It is *point transitive* if there exists $x \in X$ such that $\overline{Orb(x, T)} = X$; such x is called a *transitive point*. In our context these two notions coincide and the collection of transitive points forms a dense G_δ set in X. One says (X, T) is *weakly mixing* if the product system $(X^2, T^{(2)})$ is transitive. A TDS (X, T) is *minimal* if $\overline{Orb(x, T)} = X$ for every $x \in X$. A point $x \in X$ is *minimal* or *almost periodic* if the subsystem $(\overline{Orb(x, T)}, T)$ is minimal. The system (X, T) is *semi-simple* if every point $x \in X$ is minimal. If (X,T) is minimal then $(X^n, T^{(n)})$ has dense minimal points.

A *factor map* $\pi : X \to Y$ between the TDS (X,T) and (Y,S) is a continuous onto map which intertwines the actions; one says that (Y,S) is a *factor* of (X,T) and that (X,T) is an *extension* of (Y,S), and one refers to π as a factor or an extension. The systems are said to be *conjugate* if π is bijective.

Given a group G one says it acts on X (by homeomorphisms) if for any $g \in G$ there is a homeomorphism $\pi_g : X \to X$ (that is commonly denoted g) such that $\pi_{gh} = \pi_g \circ \pi_h$ and $\pi_1 = id$, where $g, h \in G$ and 1 is the unit of G. This group action is denoted by (X,G). An analogous definition can be given if G is a semigroup. Also, the notions of transitivity, minimality and factor are naturally generalized to group actions.

2.2. **Proximal, distal and regionally proximal relations.** Let (X,T) be a TDS. Fix $(x,y) \in X^2$. It is a *proximal* pair if $\inf_{n\in\mathbb{Z}} d(T^n x, T^n y) = 0$; it is a *distal* pair if it is not proximal. Denote by $P(X,T)$ and $D(X,T)$ the sets of proximal and distal pairs of (X,T) respectively. They are also called the proximal and distal relations. It is easy to see that

$$P(X,T) = \bigcap_{k\geq 1} \bigcup_{n\in\mathbb{Z}} T^{-n}\Delta_{\frac{1}{k}},$$

where $\Delta_\epsilon = \{(x,y) \in X^2 : d(x,y) \leq \epsilon\}$ for any $\epsilon > 0$.

A TDS (X,T) is *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T^n x, T^n y) < \epsilon$ for every $n \in \mathbb{Z}$. It is *distal* if $D(X,T) = X^2 \setminus \Delta(X)$. Any equicontinuous system is distal.

The *regionally proximal relation* of (X,T) is defined by

$$Q(X,T) = \bigcap_{k\geq 1} \overline{\bigcup_{n\in\mathbb{Z}} T^{-n}\Delta_{\frac{1}{k}}}.$$

It follows that $(x,y) \in Q(X,T)$ if and only if for any $\epsilon > 0$ and neighborhoods U, V of x, y respectively there are $x' \in U$, $y' \in V$ and $m \in \mathbb{Z}$ such that $d(T^m x, T^m y) \leq \epsilon$. It holds that (X,T) is equicontinuous if and only if $Q(X,T) = \Delta(X)$ and (X,T) is distal if and only if $P(X,T) = \Delta(X)$.

We will need the following generalization of the regionally proximal relation. Let $n \geq 2$ and $x_1, \ldots, x_n \in X$. One says $(x_1, \ldots, x_n)$ is *n-regionally proximal* if and only if for any $\epsilon > 0$ and neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ respectively, there are $x_i' \in U_i$, $i \in \{1, \ldots, n\}$, and $m \in \mathbb{Z}$ such that $diam\{T^m x_1', \ldots, T^m x_n'\} \leq \epsilon$. Denote by $Q_n(X,T)$ the set of n-regionally proximal tuples. It can be proved that

$$Q_n(X,T) = \bigcap \{ \overline{\bigcup_{m\in\mathbb{Z}} T^{-m}\alpha} : \alpha \text{ is a neighborhood of the diagonal in } X^n \}.$$

Hence it is a closed invariant set of X^n. Let $Q_n^+(X,T)$ be the set of n-regionally proximal tuples defined using $m \in \mathbb{Z}_+$ and $Q_n^-(X,T) = Q_n^+(X,T^{-1})$. It holds that $Q_n(X,T) = Q_n^+(X,T) = Q_n^-(X,T)$ (see Proposition 3.7).

2.3. **Some facts about universal minimal actions.** Let $\beta\mathbb{Z}$ be the Stone-Cech compactification of $\mathbb{Z}$, which is a compact Hausdorff topological space where $\mathbb{Z}$ is densely and equivariantly embedded. Moreover, the addition on $\mathbb{Z}$ can be extended to an addition on $\beta\mathbb{Z}$ in such a way that $\beta\mathbb{Z}$ is a closed semigroup with continuous right translations. The action of $\mathbb{Z}$ on $\beta\mathbb{Z}$ is point transitive.

Let $(M,\mathbb{Z})$ be the universal minimal action defined from $\mathbb{Z}$. The set M is a closed semigroup with continuous right translations, isomorphic to any minimal left ideal in $\beta\mathbb{Z}$. In what follows we identify M with one of such ideals. By the Ellis-Namakura Theorem (see Chapter 6, Lemma 6, in [Au]) the set $J := J(M)$ of idempotents in M is nonempty. Moreover, $\{vM : v \in J\}$ is a partition of M and every vM is a group with unit element v.

Assume $\mathbb{Z}$ acts on the compact metric space X. That is, there exists a homeomorphism $T : X \to X$ such that (X,T) is a TDS and for any $m \in \mathbb{Z}$ and $x \in X$ one has $mx = T^m x$. Then the sets $\beta\mathbb{Z}$ and M also act on X as semigroups and $\beta\mathbb{Z}x = \{px : p \in \beta\mathbb{Z}\} = \overline{Orb(x,T)}$. If the action of $\mathbb{Z}$ on X is minimal (or (X,T) is minimal) one has $Mx = \overline{Orb(x,T)} = X$ for every $x \in X$. For $x \in X$ define $J_x = \{v \in J : vx = x\}$. It holds that x is minimal if and only if J_x is not empty. Observe that for any invariant closed subset A of X, JA is the collection of minimal points in A.

Let 2^X be the collection of nonempty closed subsets of X endowed with the Hausdorff topology. Remark that a basis for this topology on 2^X is given by

$$< U_1, \dots, U_n >= \{A \in 2^X : A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for every } i \in \{1, \dots, n\}\},$$

where each $U_i \subseteq X$ is open. The action of $\mathbb{Z}$ on 2^X is given by $mA = \{ma : a \in A\}$ for each $m \in \mathbb{Z}$ and $A \in 2^X$. This action induces another one of $\beta\mathbb{Z}$ on 2^X. To avoid ambiguities one denotes the action of $\beta\mathbb{Z}$ on 2^X by the *circle operation* as follows: let $p \in \beta\mathbb{Z}$ and $A \in 2^X$, then define $p \circ A = \lim_{\lambda} m_\lambda A$ for any net $(m_\lambda : \lambda \in \Lambda)$ converging to p. Moreover

$$p \circ A = \{x \in X : \text{for each } \lambda \in \Lambda \text{ there is } d_\lambda \in A \text{ with } x = \lim_{\lambda} m_\lambda d_\lambda\}$$

for any fixed net $(m_\lambda : \lambda \in \Lambda)$ converging to p. Observe that $pA \subseteq p \circ A$, where $pA = \{pa : a \in A\}$.

2.4. **Fundamental extensions.** Let (X,T) and (Y,S) be TDS and $\pi : X \to Y$ a factor map. One says that:

- π is an *open* extension if it is open as a map;
- π is a *semi-open* extension if the image of every nonempty open set of X has nonempty interior;
- π is a *proximal* extension if $\pi(x_1) = \pi(x_2)$ implies $(x_1,x_2) \in P(X,T)$;
- π is a *distal* extension if $\pi(x_1) = \pi(x_2)$ implies $(x_1,x_2) \in D(X,T)$;
- π is an *almost one to one* extension if there exists a dense G_δ set $X_0 \subseteq X$ such that $\pi^{-1}(\{\pi(x)\}) = \{x\}$ for any $x \in X_0$;

• π is an *equicontinuous or isometric* extension if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\pi(x_1) = \pi(x_2)$ and $d(x_1, x_2) < \delta$ imply $d(T^n(x_1), T^n(x_2)) < \epsilon$ for any $n \in \mathbb{Z}$;

• π is a *group* extension if there exists a compact Hausdorff topological group K such that the following conditions hold:

(1) K acts continuously on X from the right: the right action $X \times K \to X$, $(x, k) \mapsto xk$ is continuous and $T^n(xk) = (T^n x)k$ for any $n \in \mathbb{Z}$ and $k \in K$;

(2) the fibers of π are the $K-$orbits in X: $\pi^{-1}(\{\pi(x)\}) = xK$ for any $x \in X$;

• π is a *n-weak mixing* extension for some $n \geq 2$ if the system $(R^n_\pi, T^{(n)})$ is topologically transitive, where

$$R^n_\pi = \{(x_1, \dots, x_n) \in X^n : \pi(x_1) = \dots = \pi(x_n)\}.$$

If π is n-weak mixing for any $n \geq 2$ then π is said to be *totally weakly mixing*.

We will use the following results (see Chapter VI, Section 3, [Vr]).

Theorem 2.1. *Given a factor map $\pi : X \to Y$ between minimal systems (X, T) and (Y, S) there exists a commutative diagram of factor maps (called* **O-diagram***)*

$$\begin{array}{ccc} X & \xleftarrow{\sigma^*} & X^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ Y & \xleftarrow{\tau^*} & Y^* \end{array}$$

such that
(a) σ^* and τ^* are almost one to one extensions;
(b) π^* is an open extension;
(c) X^* is the unique minimal set in $R_{\pi\tau^*} = \{(x, y) \in X \times Y^* : \pi(x) = \tau^*(y)\}$ and σ^* and π^* are the restrictions to X^* of the projections of $X \times Y^*$ onto X and Y^* respectively.

We sketch the construction of these factors. Let $x \in X$, $u \in J_x$ and $y = \pi(x)$. Let $y^* = u \circ \pi^{-1}(\{y\})$ and define $Y^* = \{p \circ y^* : p \in M\}$ as the orbit closure of y^* in 2^X for the action of $\mathbb{Z}$; one has that y^* is a minimal point so Y^* is minimal. Finally $X^* = \{(px, p \circ y^*) \in X \times Y^* : p \in M\}$, $\tau^*(p \circ y^*) = py$ and $\sigma^*((px, p \circ y^*)) = px$. It can be proved that $X^* = \{(\tilde{x}, \tilde{y}) \in X \times Y^* : \tilde{x} \in \tilde{y}\}$.

Definition 2.2. A factor map $\pi : X \to Y$ between minimal systems (X, T) and (Y, S) is said to be *RIC (relatively incontractible)* if for any $y \in Y$ and any $u \in J_y$, $\pi^{-1}(\{y\}) = u \circ u\pi^{-1}(\{y\})$. It is equivalent to $\pi^{-1}(\{py\}) = p \circ u\pi^{-1}(\{y\})$ for any $p \in M$, where $y \in Y$ and $u \in J_y$ are fixed.

It is well known that a distal extension is RIC and that a RIC extension is open. Every factor map between minimal systems can be lifted to a RIC extension by proximal extensions (see Chapter VI, Section 2, [Vr]).

Theorem 2.3. *Given a factor map $\pi : X \to Y$ between minimal systems (X, T) and (Y, S) there exists a commutative diagram of factor maps (called* **RIC-diagram***)*

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma'\ } & X' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{\ \tau'\ } & Y' \end{array}$$

such that
(a) σ' and τ' are proximal extensions;
(b) π' is a RIC extension;
(c) X' is the unique minimal set in $R_{\pi\tau'} = \{(x,y) \in X \times Y' : \pi(x) = \tau'(y)\}$ and σ' and π' are the restrictions to X' of the projections of $X \times Y'$ onto X and Y' respectively.

This construction is similar to that of the O-diagram. Let $x \in X$, $u \in J_x$ and $y = \pi(x)$. Let $y' = u \circ u\pi^{-1}(\{y\})$, then y' is a minimal point in 2^X for the action of $\mathbb{Z}$. Define $Y' = \{p \circ uy' : p \in M\}$ to be the orbit closure of y' and $X' = \{(px, p \circ y') \in X \times Y' : p \in M\}$, and factor maps given by $\tau'(p \circ y') = py$ and $\sigma'((px, p \circ y')) = px$. It can be proved that $X' = \{(\tilde{x}, \tilde{y}) \in X \times Y' : \tilde{x} \in \tilde{y}\}$.

We recall the structure theorem for minimal systems (see [EGS, Au, G1, Vr] for details).

Theorem 2.4 (Structure theorem for minimal systems)**.** *Given a minimal TDS (X,T) there exists a countable ordinal η and a canonically defined commutative diagram of minimal systems (called* **PI-tower***):*

$$\begin{array}{ccccccccccccc}
X = X_0 & \xleftarrow{\ \phi_1\ } & & & X_1 & \leftarrow & \cdots\ X_{\nu-1} & \xleftarrow{\ \phi_\nu\ } & & & X_\nu & \cdots \leftarrow & X_\eta = X_\infty \\
\Big\downarrow{\scriptstyle \pi_0} & \searrow{\scriptstyle \sigma_1} & & & \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_{\nu-1}} & \searrow{\scriptstyle \sigma_\nu} & & & \Big\downarrow{\scriptstyle \pi_\nu} & & \Big\downarrow{\scriptstyle \pi_\infty} \\
Y_0 = \{y_0\} & \xleftarrow{\ \rho_1\ } & Z_1 & \xleftarrow{\ \psi_1\ } & Y_1 & \leftarrow & \cdots\ Y_{\nu-1} & \xleftarrow{\ \rho_\nu\ } & Z_\nu & \xleftarrow{\ \psi_\nu\ } & Y_\nu & \cdots \leftarrow & Y_\eta = Y_\infty
\end{array}$$

where for each $\nu \le \eta$, ρ_ν is equicontinuous, ϕ_ν and ψ_ν are proximal, π_ν is RIC and π_∞ is RIC and weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν, etc., are the inverse limits of $X_\lambda, Y_\lambda, \pi_\lambda$, etc., for $\lambda < \nu$.

The TDS (X,T) is said to be *strictly proximal isometric* or *strictly PI* if it can be get from the trivial system by a (countable) transfinite succession of proximal and equicontinuous extensions like Y_∞; it is said to be *proximal isometric* or *PI* if in the PI-tower π_∞ is an isomorphism, or equivalently if it is the factor of a strictly PI system by a proximal extension.

2.5. **Sequence entropy.** In the article increasing sequences of nonnegative integers are denoted by $\{0 \le t_1 < t_2 < \cdots\}$. Let (X,T) be a TDS. Consider an increasing sequence of nonnegative integers A as before and a finite open cover $\mathcal{U}$ of X. The *topological sequence entropy* of $\mathcal{U}$ with respect to (X,T) along A is

$$h_A(T,\mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log N\Big(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}\Big),$$

where $N(\bigvee_{i=1}^n T^{-t_i}\mathcal{U})$ is the minimal cardinality among all cardinalities of subcovers of $\bigvee_{i=1}^n T^{-t_i}\mathcal{U}$. Recall that for open covers $\mathcal{U}$ and $\mathcal{V}$ of X, $\mathcal{U}\bigvee\mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

The *topological sequence entropy of* (X,T) along A is $h_A(T) = \sup_{\mathcal{U}} h_A(T,\mathcal{U})$, where the supremum is taken over all finite open covers of X. If $A = \mathbb{Z}_+$ one recovers the standard topological entropy. In this case one omits the subscript $\mathbb{Z}_+$. Finally the sequence entropy of (X,T) is defined by

$$h_\infty(X,T) = \sup h_A(X,T),$$

where the supremum ranges over all increasing sequences of nonnegative integers.

By an *admissible cover* $\mathcal{U}$ of X one means that $\mathcal{U}$ is finite and if $\mathcal{U} = \{U_1,\ldots,U_n\}$ then $(\bigcup_{j\neq i} U_j)^c$ has nonempty interior for each $i \in \{1,\ldots,n\}$. Let $(x_1,\ldots,x_n) \in X^n$ and $\mathcal{U} = \{U_1,\ldots,U_n\}$ be a finite cover of X. One says $\mathcal{U}$ is an *admissible cover* with respect to $(x_1,\ldots,x_n)$ if for each U_i, $i \in \{1,\ldots,n\}$, there exists $j_i \in \{1,\ldots,n\}$ such that x_{j_i} is not in the closure of U_i.

Definition 2.5. Let (X,T) be a TDS and $n \geq 2$. An n-tuple $(x_1,\ldots,x_n) \in X^n \setminus \Delta_n(X)$ is a *sequence entropy n-tuple* (n-SET) if whenever $V_1,\ldots,V_n$ are closed mutually disjoint neighborhoods of $x_1,\ldots,x_n$ respectively, there is some increasing sequence $A \subseteq \mathbb{Z}_+$ such that the open cover $\mathcal{U} = \{V_1^c,\ldots,V_n^c\}$ has positive sequence entropy along A, i.e. $h_A(T,\mathcal{U}) > 0$.

It is easy to see that an n-tuple $(x_1,\ldots,x_n) \in X^n\setminus\Delta_n(X)$ is an n-SET if and only if for any admissible open cover $\mathcal{U}$ with respect to $(x_1,\ldots,x_n)$ one has $h_A(T,\mathcal{U}) > 0$ for some increasing sequence $A \subseteq \mathbb{Z}_+$.

For $n \geq 2$ one denotes by $SE_n(X,T)$ the set of n-SET. In the case $n = 2$ one speaks about pairs instead of tuples and one writes $SE(X,T)$. The proof of the following result is analogous to the corresponding one in [B] (see Propositions 2, 3, 4 and 5 respectively).

Proposition 2.6. *Let (X,T) be a TDS and $n \geq 2$.*

1. *If $\mathcal{U} = \{U_1,\ldots,U_n\}$ is an admissible open cover of X with $h_A(T,\mathcal{U}) > 0$ for some increasing sequence $A \subseteq \mathbb{Z}_+$, then for each $i \in \{1,\ldots,n\}$ there exists $x_i \in U_i^c$ such that $(x_1,\ldots,x_n)$ is an n-SET.*
2. *$SE_n(X,T) \cup \Delta_n(X)$ is a nonempty closed $T^{(n)}$-invariant subset of X^n.*
3. *Let $\pi : X \to Y$ be a factor map of the TDS (X,T) and (Y,S).*
   (a) *If $(y_1,\ldots,y_n) \in SE_n(Y,S)$ then there exists $(x_1,\ldots,x_n) \in SE_n(X,T)$ such that $\pi(x_i) = y_i$ for $i \in \{1,\ldots,n\}$.*
   (b) *Conversely if $(x_1,\ldots,x_n) \in SE_n(X,T)$ and $(\pi(x_1),\ldots,\pi(x_n)) \notin \Delta_n(Y)$, then $(\pi(x_1),\ldots,\pi(x_n)) \in SE_n(Y,S)$.*
4. *Let W be a closed T-invariant subset of (X,T). If $(x_1,\ldots,x_n)$ is an n-SET of $(W,T|_W)$, then it is also a n-SET of (X,T).*

Definition 2.7. Let (X,T) be a TDS. It is *null* if $h_\infty(X,T) = 0$, it is *bounded* if $h_\infty(X,T) < \infty$ and it is *unbounded* if $h_\infty(X,T) = \infty$.

By Proposition 2.6 one knows that a system (X,T) is null if and only if $SE(X,T) = \emptyset$.

3. Regionally proximal relation and sequence entropy

It is not easy to decide whether there exist sequence entropy tuples in a system. In this section we try to give some general conditions which imply their existence. The results are inspired by [HLSY] where the case $n = 2$ was developed. Also we describe the relation between sequence and regionally proximally n-tuples.

Definition 3.1. Let (X, T) be a TDS. One says $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$ is a *weak mixing n-tuple* if for any open neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ respectively, $\bigcap_{i=1}^{n} N(U_1, U_i) \neq \emptyset$, where $N(U, V) = \{k \in \mathbb{Z}_+ : U \cap T^{-k}V \neq \emptyset\}$. The set of weak mixing n-tuples is denoted by $WM_n(X, T)$.

Sequence entropy, regionally proximal and weak mixing tuples are related as is shown by the following lemma.

Lemma 3.2. *Let (X, T) be a TDS and $n \geq 2$, then*

$$SE_n(X, T) \subseteq WM_n(X, T) \subseteq Q_n^-(X, T).$$

Proof. To prove the first inclusion we follow [BHM]. Assume that $(x_1, \ldots, x_n) \notin \Delta_n(X) \cup WM_n(X, T)$. Then for each $i \in \{1, \ldots, n\}$ there exists an open neighborhood U_i' of x_i such that $\bigcap_{i=1}^n N(U_1', U_i') = \emptyset$. For each $i \in \{1, \ldots, n\}$ take a closed neighborhood U_i of x_i with $U_i \subseteq U_i'$ such that $\bigcap_{i=1}^n U_i = \emptyset$. Clearly, $\bigcap_{i=1}^n N(U_1, U_i) = \emptyset$. Therefore, for each $m \in \mathbb{Z}_+$ there exists $i_m \in \{1, \ldots, n\}$ such that $U_1 \cap T^{-m} U_{i_m} = \emptyset$. Thus, if $W_m = U_{i_m}^c$ then $U_1 \subseteq T^{-m} W_m$.

Put $\mathcal{R} = \{U_1^c, \ldots, U_n^c\}$ and let $m \in \mathbb{Z}_+$. For any nonnegative integer sequence $A = \{0 \leq t_1 < t_2 < \ldots\}$ and $x \in X$ consider (if it exists) the first $i \in \{1, \ldots, m\}$ such that $T^{t_i} x \in U_1$. One gets that $\mathcal{R}_m = \bigvee_{i=1}^{m} T^{-t_i} \mathcal{R}$ admits a subcover by the sets

$$T^{-t_1} U_1^c \cap \ldots \cap T^{-t_{i-1}} U_1^c \cap T^{-t_i} W_0 \cap T^{-t_{i+1}} W_{t_{i+1} - t_i} \cap \ldots \cap T^{-t_m} W_{t_m - t_i},$$

for $i \in \{1, \ldots, m+1\}$. Hence $N(\mathcal{R}_m) \leq m + 1$ and therefore $h_A(T, \mathcal{R}) = 0$. This implies that $(x_1, \ldots, x_n) \notin SE_n(X, T)$ and thus $SE_n(X, T) \subseteq WM_n(X, T)$.

Now let $(x_1, \ldots, x_n) \in WM_n(X, T)$. Then for any open neighborhood U_i of x_i, $i \in \{1, \ldots, n\}$, one has that $\bigcap_{i=1}^n N(U_1, U_i) \neq \emptyset$. Thus there exist $m \in \mathbb{Z}_+$ and $x_i' \in U_i$ with $T^{-m} x_i' \in U_1$ for each $i \in \{1, \ldots, n\}$, which implies that $(x_1, \ldots, x_n) \in Q_n^-(X, T) \setminus \Delta_n(X)$. □

In the following we give some conditions under which a weak mixing tuple is a sequence entropy tuple.

Lemma 3.3. *Let (X, T) be a TDS and $n \geq 2$. Suppose $(x_1, \ldots, x_n) \in \overline{WM_n(X, T)}$ and $\pi_1 : Z \to X$ is semi-open where $Z = \overline{\{(T^{(n)})^m (x_1, \ldots, x_n) : m \in \mathbb{Z}_+\}}$. Let $U_1, \ldots, U_n$ be neighborhoods of $x_1, \ldots, x_n$ respectively. Then, there exists a sequence $0 \leq t_1 < t_2 < t_3 < \cdots$ in $\mathbb{Z}_+$ such that for any $m > 0$ and $s \in \{1, \ldots, n\}^m$ one can find $M_s \in \mathbb{N}$ with $T^{M_s}(x_1) \in \bigcap_{i=1}^m T^{-t_i} U_{s(i)}$ and $T^{M_s}(x_j) \in U_j$, for $j \in \{1, \ldots, n\}$.*

Proof. We can assume (X, T) is not periodic.

Since $\pi_1 : Z \to X$ is semi-open, $W_1 = int(\pi_1((U_1 \times \ldots \times U_n) \cap Z))$ is a nonempty open set of X and $W_1 \subseteq U_1$. Let $W_i = U_i$ for $i \in \{2, \ldots, m\}$.

Since $(W_1 \times \cdots \times W_n) \cap Z$ is a nonempty open set of Z and the orbit of $(x_1, \ldots, x_n)$ is dense in Z, there exists $t \in \mathbb{Z}_+$ such that $(T^t x_1, \ldots, T^t x_n) \in (W_1 \times \cdots \times W_n) \cap Z$. From the fact $(x_1, \ldots, x_n) \in WM_n(X, T)$ it follows that $(T^t x_1, \ldots, T^t x_n) \in WM_n(X, T)$. Therefore, $\bigcap_{i=1}^n N(W_1, W_i) \neq \emptyset$ and there exists $t_1 \geq 0$ such that $W_1 \cap T^{-t_1} W_i \neq \emptyset$ for $i \in \{1, \ldots, n\}$. Since the orbit of $(x_1, \ldots, x_n)$ is dense in Z, there exists $M_i \in \mathbb{Z}_+$ such that $(T^{M_i} x_1, \ldots, T^{M_i} x_n) \in ((W_1 \cap T^{-t_1} W_i) \times W_2 \times \cdots \times W_n) \cap Z$, for $i \in \{1, \ldots, n\}$. One concludes that $T^{M_i}(x_1) \in U_1 \cap T^{-t_1} U_i$ and $T^{M_i}(x_j) \in U_j$ for $i, j \in \{1, \ldots, n\}$.

Now suppose $0 \leq t_1 < t_2 < \ldots < t_m$, $m \geq 1$, have been defined and satisfy that for any $s \in \{1, \ldots, n\}^m$ there exists $M_s \in \mathbb{N}$ such that $T^{M_s}(x_1) \in \bigcap_{i=1}^m T^{-t_i} U_{s(i)}$ and $T^{M_s}(x_j) \in U_j$ for $j \in \{1, \ldots, n\}$.

We are going to define t_{m+1}. Take $\delta > 0$ such that for any $z_i \in X$ with $d(z_i, x_i) < \delta$, $1 \leq i \leq n$, one has $T^{M_s}(z_1) \in \bigcap_{i=1}^m T^{-t_i} U_{s(i)}$ and $T^{M_s}(z_j) \in U_j$ for $j \in \{1, \ldots, n\}$ and any $s \in \{1, \ldots, n\}^m$. Let $U_i^\delta = \{z \in X : d(z, x_i) < \delta\}$.

Since $\pi_1 : Z \to X$ is semi-open, $W_1^\delta = int(\pi_1((U_1^\delta \times \ldots \times U_n^\delta) \cap Z))$ is a nonempty open set and $W_1^\delta \subseteq U_1^\delta$. Let $W_i^\delta = U_i^\delta$ for $i \in \{2, \ldots, n\}$. By the same argument as above one has $\bigcap_{i=1}^n N(W_1^\delta, W_i^\delta) \neq \emptyset$. Without loss of generality (taking δ as small as necessary) one can assume that $\bigcap_{i=1}^n N(W_1^\delta, W_i^\delta) \subseteq \{t_m + 1, t_m + 2, \ldots\}$. Thus there exists $t_{m+1} > t_m$ such that $W_1^\delta \cap T^{-t_{m+1}} W_i^\delta \neq \emptyset$, $i \in \{1, \ldots, n\}$. One can choose $P_i \in \mathbb{Z}_+$ such that $T^{P_i}(x_1) \in W_1^\delta \cap T^{-t_{m+1}} W_i^\delta$ and $T^{P_i}(x_j) \in W_j^\delta$ for $i, j \in \{1, \ldots, n\}$.

For any $r \in \{1, \ldots, n\}^{m+1}$ let $s \in \{1, \ldots, n\}^m$ with $s(i) = r(i)$, $i \in \{1, \ldots, m\}$, and let $r(m+1) = k$. Put $M_r = M_s + P_k$. Then

$$T^{M_r}(x_1) = T^{M_s}(T^{P_k}(x_1)) \in \bigcap_{i=1}^m T^{-t_i} U_{s(i)}.$$

Since $T^{t_{m+1}} T^{P_k}(x_1) \in W_k^\delta$, then $T^{t_{m+1}} T^{M_r}(x_1) = T^{M_s} T^{t_{m+1}} T^{P_k}(x_1) \in U_k$. Thus

$$T^{M_r}(x_1) \in (\bigcap_{i=1}^m T^{-t_i} U_{s(i)}) \bigcap (T^{-t_{m+1}} U_k) = \bigcap_{i=1}^{m+1} T^{-t_i} U_{r(i)}.$$

Moreover, $T^{M_r}(x_j) = T^{M_s} T^{P_k}(x_j) \in U_j$. The proof of the lemma is completed. $\square$

Lemma 3.4. *Let (X, T) be a TDS and $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$. If for any neighborhood U_i of x_i for $i \in \{1, \ldots, n\}$ there is an increasing sequence $A = \{0 \leq t_1 < t_2 < \ldots\}$ in $\mathbb{Z}_+$ such that for any $m \in \mathbb{N}$ and $s = (s(1), \ldots, s(m)) \in \{1, \ldots, n\}^m$, $\bigcap_{i=1}^m T^{-t_i} U_{s(i)} \neq \emptyset$, then*

$$(x_1, \ldots, x_n) \in SE_n(X, T) \quad \text{and} \quad h_\infty(T) \geq logN,$$

where N is the cardinality of the set $\{x_1, \ldots, x_n\}$. In particular, if $(x_1, \ldots, x_n) \notin \Delta^{(n)}(X)$ one gets $h_\infty(X) \geq \log n$.

Proof. Without loss of generality one can assume $x_i \neq x_j$ for $i \neq j$. Let $U_1, \ldots, U_n$ be closed mutually disjoint neighborhoods of $x_1, \ldots, x_n$ respectively. By assumption, one knows there exists an increasing sequence $A = \{0 \le t_1 < t_2 < t_3 < \ldots\}$ such that for any $m > 0$ and $s \in \{1, \ldots, n\}^m$ one can find $x_s \in \bigcap_{i=1}^{m} T^{-t_i} U_{s(i)}$.

Let $X_m = \{x_s : s \in \{1, \ldots, n\}^m\}$. Remark that for every $s \in \{1, \ldots, n\}^m$ one has $\#(\bigcap_{i=1}^{m} T^{-t_i} U_{s(i)}^c \cap X_m) = (n-1)^m$. Combining this fact with $\#(X_m) = n^m$ one gets $N(\bigvee_{i=1}^{m} T^{-t_i} \mathcal{U}) \ge (\frac{n}{n-1})^m$, where $\mathcal{U} = \{U_1^c, \ldots, U_n^c\}$.

Hence $h_A(T, \mathcal{U}) \ge \limsup_{m \to \infty} \frac{1}{m} \log N(\bigvee_{i=1}^{m} T^{-t_i} \mathcal{U}) \ge \log(\frac{n}{n-1})$ and thus $(x_1, \ldots, x_n) \in SE_n(X, T)$.

Define $V_j = X \setminus (\bigcup_{i \ne j} U_i), 1 \le j \le n$, and $\mathcal{V} = \{V_1, \ldots, V_n\}$. Then $\mathcal{V}$ is an admissible open cover of X. Observe that for every $s \in \{1, \ldots, n\}^m$ one has $\#(\bigcap_{i=1}^{m} T^{-t_i} V_{s(i)} \cap X_m) = 1$. This fact and $\#(X_m) = n^m$ implies $N(\bigvee_{i=1}^{m} T^{-t_i} \mathcal{V}) = n^m$. Hence

$$h_\infty(T) \ge h_A(T, \mathcal{V}) \ge \limsup_{m \to \infty} \frac{1}{m} \log N(\bigvee_{i=1}^{m} T^{-t_i} \mathcal{V}) = \log n.$$

□

We conclude the following useful lemma.

Lemma 3.5. *Let (X, T) be a TDS, $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$ and $Z = \overline{\{(T^{(n)})^m (x_1, \ldots, x_n) : m \in \mathbb{Z}_+\}}$. Consider the projection $\pi_1 : Z \to X$ to the first coordinate. If π_1 is semi-open then $(x_1, \ldots, x_n) \in SE_n(X, T)$ if and only if $(x_1, \ldots, x_n) \in WM_n(X, T)$.*

Proof. By Lemma 3.2, one knows $SE_n(X, T) \subseteq WM_n(X, T)$. Suppose $(x_1, \ldots, x_n) \in WM_n(X, T)$ and $\pi_1 : Z \to X$ is semi-open. One shows that $(x_1, \ldots, x_n) \in SE_n(X, T)$ from Lemmata 3.4 and 3.3. □

Since a homomorphism between minimal systems is semi-open, one gets the following corollary.

Corollary 3.6. *Let (X, T) be a minimal system and $(x_1, \ldots, x_n) \in X^n$. If $(x_1, \ldots, x_n)$ is a minimal point, then $(x_1, \ldots, x_n) \in SE_n(X, T)$ if and only if $(x_1, \ldots, x_n) \in WM_n(X, T)$.*

Proposition 3.7. *Let (X, T) be a minimal system and $n \ge 2$. Then*

$$WM_n(X, T) = Q_n(X, T) \setminus \Delta_n(X) = Q_n^+(X, T) \setminus \Delta_n(X) = Q_n^-(X, T) \setminus \Delta_n(X).$$

Proof. First we show $Q_n^+(X, T) = Q_n^-(X, T)$. Let $(x_1, \ldots, x_n) \in Q_n^-(X, T)$. For each $\epsilon > 0$ and neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ respectively, there exist $(x_1', \ldots, x_n') \in U_1 \times \cdots \times U_n$ and $m \in \mathbb{Z}_+$ such that $diam\{T^{-m} x_1', \ldots, T^{-m} x_n'\} < \epsilon/3$. Since $(X^n, T^{(n)})$ has dense minimal points, one can assume that $(x_1', \ldots, x_n')$ is one of them. Hence there is $l \in \mathbb{Z}_+$ such that $d(T^l x_i', T^{-m} x_i') < \epsilon/3$ for each $i \in \{1, \ldots, n\}$. Thus $diam\{T^l x_1', \ldots, T^l x_n'\} < \epsilon$. That is $(x_1, \ldots, x_n) \in Q_n^+(X, T)$.

Similarly one has $Q_n^+(X,T) \subseteq Q_n^-(X,T)$ and so $Q_n^+(X,T) = Q_n^-(X,T)$. The proof for $Q_n(X,T) = Q_n^+(X,T)$ follows the same lines.

From Lemma 3.2 one has that $WM_n(X,T) \subseteq Q_n^-(X,T)$, thus it remains to show that

$$WM_n(X,T) \supseteq Q_n^-(X,T) \setminus \Delta_n(X).$$

To each $(x_1,\dots,x_n) \in Q_n^-(X,T)$ associate the subset $L(x_1,\dots,x_n)$ of X such that, for each x_0 in it and neighborhoods U_i of x_i for $i \in \{0,1,\dots,n\}$, there are $x'_1,\dots,x'_n \in U_0$ and $m \in \mathbb{Z}_+$ with $T^m x'_j \in U_j$ for $j \in \{1,\dots,n\}$. It is easy to verify that $L(x_1,\dots,x_n)$ is nonempty, closed and invariant. Also the following property holds: $(x_1,\dots,x_n) \in WM_n(X,T)$ if and only if $x_1 \in L(x_1,\dots,x_n)$. Since (X,T) is minimal, one has $X = L(x_1,\dots,x_n)$. In particular, $x_1 \in L(x_1,\dots,x_n)$. Thus $(x_1,\dots,x_n) \in WM_n(X,T)$. □

The next two theorems are the main results of the section.

Theorem 3.8. *Let (X,T) be a minimal system and $n \geq 2$. Then any minimal point in $Q_n(X,T) \setminus \Delta_n(X)$ is a sequence entropy tuple, that is,*

$$JQ_n(X,T) \setminus \Delta_n(X) \subseteq SE_n(X,T).$$

Proof. It follows from Proposition 3.7 and Corollary 3.6. □

Now we give a condition to verify whether the set $JQ_n(X,T) \setminus \Delta^{(n)}(X)$ is nonempty. Before that we recall some properties of minimal points. For $x \in X$ and $U \subseteq X$ set $N(x,U) = \{n \in \mathbb{Z} : T^n x \in U\}$. A sequence $F \subseteq \mathbb{Z}$ is *syndetic* if there is $L \in \mathbb{N}$ such that $\{i,\dots,i+L-1\} \cap F \neq \emptyset$ for every $i \in \mathbb{Z}$; L is said to be a *gap* of F. Gottschalk-Hedlund Theorem says that $x \in X$ is a minimal point if and only if for any neighborhood U of x, $N(x,U)$ is a syndetic set [Au].

Theorem 3.9. *Let (X,T) be a minimal system and $x_1,\dots,x_n \in X$. If $(x_1,\dots,x_n)$ is minimal and $(x_i,x_{i+1}) \in Q(X,T)$ for $1 \leq i \leq n-1$, then $(x_1,\dots,x_n) \in Q_n(X,T)$.*

Proof. If $n = 2$ the result is obvious. One assumes the result holds for $n \geq 2$ and we show it holds for $n+1$.

Let $(x_1,\dots,x_{n+1})$ be a minimal point in X^{n+1}. Then, there is an idempotent $u \in J$ such that $u(x_1,\dots,x_{n+1}) = (x_1,\dots,x_{n+1})$. We show that for any $\epsilon > 0$ and open neighborhoods U_i of x_i for $i \in \{1,\dots,n+1\}$, there are $m \in \mathbb{Z}$ and $(\tilde{x}_1,\dots,\tilde{x}_{n+1}) \in U_1 \times \cdots \times U_{n+1}$ such that $diam\{T^m\tilde{x}_1,\dots,T^m\tilde{x}_{n+1}\} < \epsilon$.

Let $W = \{\xi \in M : \xi x_i \in U_i, i \in \{1,\dots,n+1\}\}$, where M is the universal minimal system defined in subsection 2.3. Since $u \in W$, one gets W is a nonempty open set of M. Also, since the map $M \to X$, $p \mapsto px_n$ is semi-open, $V = int(Wx_n) \neq \emptyset$.

Choose $x'_n \in V$. From definition of V, there is $\xi_1 \in W$ such that $x'_n = \xi_1 x_n$. Put $x'_i = \xi_1 x_i$ for each $i \in \{1,\dots,n\}$; clearly $x'_i \in U_i$. Since $(x'_1,\dots,x'_n) \in \xi_1 Q_n(X) \subseteq Q_n(X)$ and the minimal points for $(X^n, T^{(n)})$ are dense, then there exists a minimal point $(x''_1,\dots,x''_n) \in U_1 \times \cdots \times U_{n-1} \times V$ such that

$$N((x''_1,\dots,x''_n), \Delta^{(n)}_{\epsilon/2}) \neq \emptyset, \tag{1}$$

where $\Delta_\epsilon^{(n)} = \{(z_1, \dots, z_n) \in X^n : diam\{z_1, \dots, z_n\} < \epsilon\}$ is a neighborhood of the diagonal of X^n. Moreover, the minimality of $(x_1'', \dots, x_n'')$ implies $N((x_1'', \dots, x_n''), \Delta_{\epsilon/2}^{(n)})$ is syndetic. Let L be a gap of $N((x_1'', \dots, x_n''), \Delta_{\epsilon/2}^{(n)})$ and $\delta > 0$ be such that for any $x, y \in X$ with $d(x, y) < \delta$ one has that $d(T^m x, T^m y) < \epsilon/2$ for $m \in \{1, \dots, L\}$.

As before, since $(x_1'', \dots, x_n'')$ is minimal, there is $v \in J$ such that $v(x_1'', \dots, x_n'') = (x_1'', \dots, x_n'')$. Let $\widetilde{W} = \{\xi \in M : \xi x_1'' \in U_1, \dots, \xi x_{n-1}'' \in U_{n-1}, \xi x_n'' \in V\}$. Since $v \in \widetilde{W}$ and $M \to X, p \mapsto p x_n''$, is semi-open, then $\widetilde{W}$ is nonempty and $\widetilde{V} = int(\widetilde{W} x_n'') \neq \emptyset$.

Fix x_n''' in $\widetilde{V}$. Observe that $\widetilde{V} \subseteq \widetilde{W} x_n'' \subseteq V \subseteq W x_n$, thus there is $\xi_2 \in W$ such that $x_n''' = \xi_2 x_n$. Let $x_{n+1}''' = \xi_2 x_{n+1} \in U_{n+1}$. One deduces $(x_n''', x_{n+1}''') = \xi_2(x_n, x_{n+1}) \in \xi_2 Q(X, T) \subseteq Q(X, T)$. Hence there is a minimal point $(\tilde{x}_n, \tilde{x}_{n+1}) \in \widetilde{V} \times U_{n+1}$ such that

$$N((\tilde{x}_n, \tilde{x}_{n+1}), \Delta_\delta^{(2)}) \neq \emptyset. \tag{2}$$

Since $(\tilde{x}_n, \tilde{x}_{n+1})$ is minimal, $N((\tilde{x}_n, \tilde{x}_{n+1}), \Delta_\delta^{(2)})$ is syndetic and by the definition of δ, the set

$$\{m \in \mathbb{Z} : \{m, \dots, m + L - 1\} \subseteq N((\tilde{x}_n, \tilde{x}_{n+1}), \Delta_{\epsilon/2}^{(2)})\} \tag{3}$$

is syndetic.

Since $\tilde{x}_n \in \widetilde{V} \subseteq \widetilde{W} x_n''$, there is $\xi_3 \in \widetilde{W}$ such that $\tilde{x}_n = \xi_3 x_n''$. Let $\tilde{x}_i = \xi_3 x_i'' \in U_i$, for $i \in \{1, \dots, n-1\}$. Since $(x_1'', \dots, x_n'')$ is a minimal point in X^n, then $(\tilde{x}_1, \dots, \tilde{x}_n)$ is minimal too. From (1) one gets

$$N((\tilde{x}_1, \dots, \tilde{x}_n), \Delta_{\epsilon/2}^{(n)}) \neq \emptyset, \tag{4}$$

and it is syndetic with gap L. From (3) and (4) one deduces

$$N((\tilde{x}_n, \tilde{x}_{n+1}), \Delta_{\epsilon/2}^{(2)}) \cap N((\tilde{x}_1, \dots, \tilde{x}_n), \Delta_{\epsilon/2}^{(n)}) \neq \emptyset.$$

Thus there is $m \in \mathbb{Z}$ such that $diam\{T^m \tilde{x}_1, \dots, T^m \tilde{x}_{n+1}\} < \epsilon$. That is, $(x_1, \dots, x_{n+1}) \in Q_{n+1}(X, T)$. The proof is completed. □

4. The structure of bounded systems

Let (X, T) be a minimal system. It is well know that $Q(X, T)$ is an invariant closed equivalence relation and hence this relation defines the maximal equicontinuous factor $X_{eq} = X/Q(X, T)$ of (X, T) (see Chapter 9 [Au]). If $P(X, T) = Q(X, T)$ then one says it is a *proximal equicontinuous* system. It can be checked that (X, T) is proximally equicontinuous if and only if it is a proximal extension of an equicontinuous system. If (X, T) is an almost one to one extension of some equicontinuous system then one calls it an *almost automorphic* system.

Lemma 4.1. *Let $\pi : X \to Y$ be an open and finite to one extension of the minimal systems (X, T) and (Y, S). Then it is a constant to one equicontinuous extension.*

Proof. Fix $y \in Y$ and let $\pi^{-1}(\{y\}) = \{x_1, \dots, x_n\}$. From openness of π one has that $\pi^{-1}(\{py\}) = p \circ \pi^{-1}(\{y\}) = \{px_1, \dots, px_n\}$ for any $p \in M$ and that the cardinality of the set is n. Hence, by minimality, all fibers of π have the same cardinality n, which proves the map is constant to one. Considering $v \in J$ such that $vy = y$ one deduces $\pi^{-1}(\{y\}) = \{vx_1, \ldots, vx_n\}$; and consequently π is distal.

Now we show π is equicontinuous. If this property does not hold there exists $\epsilon > 0$ such that for any $k \in \mathbb{N}$ there are $(x_k, x'_k) \in R^2_\pi$ and $n_k \in \mathbb{Z}$ with $d(x_k, x'_k) < 1/k$ and $d(T^{n_k}x_k, T^{n_k}x'_k) \geq \epsilon$. Let $y_k = \pi(x_k) = \pi(x'_k)$ and assume $y_k \to y \in Y$ as $k \to \infty$. Since π is open, $\pi^{-1}(\{y_k\}) \to \pi^{-1}(\{y\})$ in the Hausdorff topology. Thus $(x_k, x'_k) \to (x, x')$ as $k \to \infty$, where x and x' are distinct points in $\pi^{-1}(\{y\})$. This contradicts the fact $d(x_k, x'_k) < 1/k$ for all $k \in \mathbb{N}$. Thus π is equicontinuous. $\square$

Theorem 4.2. *Let (X, T) be a minimal system. If (X, T) is bounded, i.e. $h_\infty(X, T) < \infty$, then (X, T) has the following structure:*

$$\begin{array}{ccc} X & \xleftarrow{\sigma'} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ Y = X_{eq} & \xleftarrow{\tau'} & Y' \end{array}$$

where σ' and τ' are proximal extensions, π' is a finite to one equicontinuous extension and π is the maximal equicontinuous factor. Moreover, π' is an isomorphism if and only if (X, T) is proximally equicontinuous.

Proof. Let $\pi : X \to Y = X_{eq}$ be the maximal equicontinuous factor of (X, T). Fix $x_0 \in X$, $u \in J_{x_0}$ and $y_0 = \pi(x_0)$. We start with a claim.
Claim. $\#(u\pi^{-1}(\{y_0\})) < \infty$.
Proof of Claim: If $\#(u\pi^{-1}(\{y_0\})) = \infty$, one shows $h_\infty(X, T) = \infty$. Indeed, since $Q(X, T)$ is an equivalence relation, then any two points $x, y \in \pi^{-1}(\{y_0\})$ are regionally proximal. Now, since $\#(u\pi^{-1}(\{y_0\})) = \infty$, for any $n \in \mathbb{N}$ one can choose n distinct elements $x_1, \ldots, x_n$ from $u\pi^{-1}(\{y_0\})$. Remark that $(x_1, \ldots, x_n)$ is a minimal point. By Theorem 3.9, $(x_1, \ldots, x_n) \in Q_n(X, T)$. From minimality of $(x_1, \ldots, x_n)$ and Theorem 3.8, one gets that $(x_1, \ldots, x_n) \in SE_n(X, T) = WM_n(X, T)$. Then by Lemmata 3.3 and 3.4 one concludes $h_\infty(X, T) \geq \log n$. Since n is arbitrary $h_\infty(X, T) = \infty$ which is a contradiction. This proves the claim.

Let $\#(u\pi^{-1}(\{y_0\})) = n \in \mathbb{N}$. If $n = 1$, then there is no minimal point in $R^2_\pi \setminus \Delta(X)$. Hence π itself is proximal and (X, T) is proximally equicontinuous. If $n \geq 2$, consider the 'RIC-diagram':

$$\begin{array}{ccc} X & \xleftarrow{\sigma'} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{\tau'} & Y' \end{array}$$

From discussion after Theorem 2.3, $Y' = \{p \circ y' : p \in M\}$, where $y' = u \circ u\pi^{-1}(\{y_0\})$, $X' = \{(px_0, p \circ y') \in X \times Y' : p \in M\}$ and $\pi'((px_0, p \circ y')) = p \circ y'$. Since $u\pi^{-1}(\{y_0\})$ is finite, then $y' = u \circ u\pi^{-1}(\{y_0\}) = u\pi^{-1}(\{y_0\})$ is finite and consequently every element of Y' is also finite with the same cardinality n as y'.

By the definition of π', $\#(\pi'^{-1}(\{z\})) = n$, for all $z \in Y'$. Hence by Lemma 4.1, π' is an n to 1 equicontinuous extension. $\square$

We do not know if σ' and τ' can be replaced by one to one extensions. In the next we do this for a particular class of minimal systems.

Definition 4.3. Let (X,T) be a minimal system and $\pi : X \to X_{eq}$ be its maximal equicontinuous factor. Consider $x \in X$, $u \in J_x$ and $y = \pi(x)$. We say (X,T) has *finite type* if $\#(u \circ \pi^{-1}(\{y\})) < \infty$. Otherwise, we say it has *infinite type*. If (X,T) has finite type, then we set $\varrho(X) = \#(u \circ \pi^{-1}(\{y\})$.

The definition does not depend on the choice of $x \in X$ and $u \in J_x$:

Lemma 4.4. *Let (X,T) be a minimal system and $\pi : X \to X_{eq}$ be its maximal equicontinuous factor. Let $x, x' \in X$, $u \in J_x$ and $v \in J_{x'}$. Then $\#(u \circ \pi^{-1}(\{\pi(x)\})) = \#(v \circ \pi^{-1}(\{\pi(x')\}))$.*

Proof. Let $y = \pi(x)$ and $y' = \pi(x')$. By minimality, there is $p \in M$ such that $py = y'$. Then $v \circ p \circ u \circ \pi^{-1}(\{y\}) \subseteq v \circ \pi^{-1}(\{y'\})$. But points in $u \circ \pi^{-1}(\{y\})$ are distal, so $\#(u \circ \pi^{-1}(\{y\})) = \#(v \circ p \circ u \circ \pi^{-1}(\{y\})) \leq \#(v \circ \pi^{-1}(\{y'\}))$. By symmetry one deduces $\#(u \circ \pi^{-1}(\{y\})) = \#(v \circ \pi^{-1}(\{y'\}))$. □

Proposition 4.5. *Let (X,T) be a finite type minimal system. Then it has the following structure:*

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma^*\ } & X^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ Y = X_{eq} & \xleftarrow{\ \tau^*\ } & Y^* \end{array}$$

where σ^* and τ^* are almost one to one, π^* is a $\varrho(X)$ to one equicontinuous extension and π is the maximal equicontinuous factor of (X,T).

Proof. We follow the discussion after the definition of the 'O-diagram' associated to (X,T) in Section 2.

Fix $x \in X$, $y = \pi(x)$ and $u \in J_x$. Then $Y^* = \{p \circ (u \circ \pi^{-1}(\{y\})) : p \in M\}$ and $X^* = \{(\tilde{x}, \tilde{y}) \in X \times Y^* : \tilde{x} \in \tilde{y}\}$. Thus $(\pi^*)^{-1}(\{\tilde{y}\}) = \tilde{y} \times \{\tilde{y}\}$ which implies $\#(\pi^*)^{-1}(\{\tilde{y}\}) = \#\tilde{y}$. But, since π^* is open, by Lemma 4.1 one has $\#\tilde{y} = \#(u \circ \pi^{-1}(\{y\})) = \varrho(X)$. This concludes the proof. □

By Lemma 4.4 and Proposition 4.5, it is easy to see that a minimal system (X,T) has finite type if and only if there is some fiber which is finite. If $\varrho(X) = 1$, then it is just an almost one to one extension of its maximal equicontinuous factor, i.e. almost automorphic system. Then systems of finite type generalize the notion of almost automorphic systems.

Lemma 4.6. *If a minimal system (X,T) is proximally equicontinuous but not almost automorphic, then $h_\infty(X,T) \geq \log 2$.*

Proof. By hypothesis $\pi : X \to Y = X_{eq}$ is proximal but not almost one to one. Consider the 'O-diagram' of (X,T). In this situation π^* is a nontrivial open extension that inherits proximality from π. Hence π^* is weakly mixing (see [G3] or [Wo1]). This implies there is a transitive point (x_1, x_2) of $R^2_{\pi^*}$ which is also a weak-mixing pair. Now, by definition, the projection from $R^2_{\pi^*}$ to X^* is semi-open. Thus by Lemmata 3.3 and 3.4 $h_\infty(X^*, T) \geq \log 2$. Since σ^* is a projection, one gets $h_\infty(X, T) \geq \log 2$. □

As a corollary one obtains the following result first proved in [HLSY].

Theorem 4.7. *Any null minimal system is an almost one to one extension of an equicontinuous system, i.e. it is an almost automorphic system.*

Proof. Let $\pi : X \to X_{eq} = X/Q(X,T)$. If there is some minimal point in $Q(X,T) \setminus \Delta(X)$, then by Theorem 3.8 it is a sequence entropy pair. Hence there is no minimal point in $R_\pi^2 \setminus \Delta(X)$, which means π is proximal. If π is not almost one to one, then by Lemma 4.6 it is not null. This finishes the proof. □

Question 4.8. *Since a nontrivial open proximal extension is not necessarily totally weakly mixing (i.e. not all R_π^n are transitive), then with the method of Lemma 4.6 one cannot go further. We conjecture that if a minimal system (X,T) is proximally equicontinuous but not almost automorphic, then $h_\infty(X,T) = \infty$. If this is true, then one can show that any minimal system (X,T) with $h_\infty(X,T) < \infty$ has finite type.*

5. Miscellaneous results

In this section we give some conditions which imply that $h_\infty(X,T) = \infty$ and we discuss the structure of a regular minimal system via sequence entropy. One needs the following lemma.

Lemma 5.1 (Sacker-Sell)**.** [SS] *Let $\pi : X \to Y$ be an extension of distal minimal systems (X,T) and (Y,S). If there is some $y \in Y$ with $\#(\pi^{-1}(\{y\}))$ finite, then (X,T) is equicontinuous if and only if (Y,S) is.*

Theorem 5.2. *Let (X,T) be a minimal system. If it satisfies one of the following conditions, then $h_\infty(X,T) = \infty$.*

(1) *distal but not equicontinuous;*
(2) *not PI.*

Proof. (1) It is well known that (X,T) is distal if and only if any point of X^n is minimal for any $n \in \mathbb{N}$ (see Chapter 5, Theorem 6 [Au]). Let $\pi : X \to X_{eq}$ be the maximal equicontinuous factor of (X,T). Since (X,T) is not equicontinuous, by Lemma 5.1, π is nontrivial and every fiber is infinite.

For $n \geq 2$ take n distinct points $x_1, \ldots, x_n$ in the same fiber. By Theorem 3.9, $(x_1, \ldots, x_n) \in Q_n(X,T)$ and from Theorem 3.8 and Lemma 3.4 $h_\infty(X,T) \geq \log n$. Since n is arbitrary, one concludes $h_\infty(X,T) = \infty$.

(2) Assume (X,T) is not PI. Let $\pi : X \to Y = X_{eq}$, $x \in X$, $u \in J_x$ and $y = \pi(x)$. Then, either $\#(u \circ \pi^{-1}(\{y\}) < \infty$ or it has the structure given in Theorem 4.2 and become a PI system. Hence $\#(u \circ \pi^{-1}(\{y\}) = \infty$. Analogously to the proof of Theorem 4.2 one has $h_\infty(X,T) = \infty$. □

Corollary 5.3. *Let (X,T) be a minimal system. If $h_\infty(X,T) < \infty$, then (X,T) must be a PI-flow.*

When a system is regular Theorem 4.2 can be stated in a cleaner form. Let $End(X,T)$ be the set of endomorphisms and $Aut(X,T)$ be the group of automorphisms of the system (X,T)

Definition 5.4. Let $\phi : X \to Y$ be an extension of minimal systems (X,T) and (Y,S). One says ϕ is *regular* if for any point $(x_1, x_2) \in R^2_\phi$ there exists $\chi \in End(X)$ such that $(\chi(x_1), x_2) \in P(X,T)$. It is equivalent to: for any minimal point (x_1, x_2) in R^2_π there exists $\chi \in End(X,T)$ such that $\chi(x_1) = x_2$.

It can be proved that the endomorphisms in the definition may be assumed to be an automorphisms. Examples of regular extensions are proximal extensions.

Lemma 5.5 (Glasner)**.** [G2] *If (X,T) is a minimal regular system, then its maximal equicontinuous factor X_{eq} is a compact group rotation and $\pi : X \to X_{eq}$ is RIC and weakly mixing.*

Proposition 5.6. *Let (X,T) be a minimal regular system. Then the following conditions are equivalent:*
1. *(X,T) is bounded;*
2. *(X,T) is null;*
3. *(X,T) is a compact group rotation.*

Proof. It is easy to see that any compact group rotation is null, and hence bounded. Now we show that (X,T) is a compact group rotation whenever (X,T) is bounded. If not, by Lemma 5.5, $\pi : X \to X_{eq}$ is a nontrivial RIC and weakly mixing extension. This means (X,T) is not PI. One concludes by Theorem 5.2 that $h_\infty(X,T) = \infty$. A contradiction. □

To finish this section let us ask a natural question: whether in the minimal case sequence entropy $h_\infty(X,T)$ is always ∞ or *log k* for some positive integer k.

6. Examples

We start with a general fact.

Lemma 6.1. *Let $\pi : X \to Y$ be an open n to one extension of minimal systems (X,T) and (Y,S). If (Y,S) is the maximal null factor of (X,T) then $h_\infty(X,T) = \log n$.*

Proof. First we recall Proposition 2.5 in [Go]: $h_\infty(X,T) \le \log n$. From Lemma 4.1 one deduces π is a distal extension and if $\{x_1, \dots, x_n\}$ is a fiber of π then $(x_1, \dots, x_n)$ is minimal for $T^{(n)}$. Recall X_{eq} is a factor of Y. Then, from Theorems 3.8 and 3.9, one gets $h_\infty(X,T) \ge \log n$. □

We give as examples a classical family of minimal symbolic systems.

6.1. Example 1: Morse system.

Let τ be the substitution $\tau(0) = 01$ and $\tau(1) = 10$. By concatenating, this map can be defined on any finite word $w = w_0 \dots w_{l-1}$ in $\{0,1\}$: $\tau(w) = \tau(w_0) \dots \tau(w_{l-1})$. For any $n \ge 2$ define $\tau^n(w) = \tau(\tau^{n-1}(w))$. Finally define $X \subseteq \{0,1\}^{\mathbb{Z}}$ to be the set of biinfinite binary sequences x in X such that any finite word in x is a subword of $\tau^n(0)$ for some $n \in \mathbb{N}$. The dynamical system (X,T) where T is the left shift map is called Morse system. It is well known that it is minimal and has the following structure: $\pi_1 : X \to Y$ and $\pi_2 : Y \to X_{eq}$ where π_1 is a 2-to-one distal extension and π_2 is an asymptotic extension (so almost one to one). Moreover, π_1 is the maximal null factor of (X,T). Then by Lemma 6.1 one concludes $h_\infty(X,T) = \log 2$.

6.2. **Example 2: Generalized Morse system.** Let us generalize previous example to get $h_\infty(X,T) = \log n$. Consider the substitution τ on $\{0,\dots,n-1\}$: $\tau(0) = 01\dots(n-1)$, $\tau(1) = 12\dots 0$,...,$\tau(n-1) = (n-1)0\dots(n-2)$. As before, by concatenating, this map can be defined on any finite word w in $\{0,\dots,n-1\}$. Define $X \subseteq \{0,\dots,n-1\}^{\mathbb{Z}}$ to be the set of biinfinite binary sequences x in X such that any finite word in x is a subword of $\tau^n(0)$ for some $n \in \mathbb{N}$. The dynamical system (X,T) where T is the left shift map is called the n-Morse system. It is minimal and has the following structure: $\pi_1 : X \to Y$ and $\pi_2 : Y \to X_{eq}$ where π_1 is a n-to-one distal extension and π_2 is an asymptotic extension (so almost one to one). Moreover, π_1 is the maximal null factor of (X,T). Then by Lemma 6.1 one concludes $h_\infty(X,T) = \log n$.

6.3. **Example 3: Rees' example** [R]**.** First we give an alternative definition for the sequence entropy. Let (X,T) be a TDS and $A = \{t_1 < t_2 < \dots\} \subseteq \mathbb{Z}_+$. One says that a set $W \subseteq X$, (T,A,ϵ,n)-*spans* $B \subseteq X$ if for any $x \in B$ there is $y \in W$ such that $d(T^{t_i}x, T^{t_i}y) < \epsilon$ for all $1 \le i \le n$, where $\epsilon > 0$ and $n \in \mathbb{N}$. A subset of X is said to be (T,A,ϵ,n)- *spanning* if it (T,A,ϵ,n)-spans X. Let $Span(T,A,\epsilon,n)$ denotes the smallest cardinality of all (T,A,ϵ,n)-spanning sets. Then one can prove that

$$h_A(X,T) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Span(T,A,\epsilon,n).$$

For the details see [Go].

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle and $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. For $x = a + \mathbb{Z}, y = b + \mathbb{Z} \subseteq \mathbb{T}$, their distance is given by $d(x,y) = \inf_{p\in\mathbb{Z}} |a-b+p|$. If $z_1 = (x_1,y_1), z_2 = (x_2,y_2) \in \mathbb{T}^2$, let $d(z_1,z_2) = \max\{d(x_1,x_2), d(y_1,y_2)\}$.

The following example was given by M. Rees. We are going to show that it is null. Let $Y = \mathbb{T}^2$ and $S : Y \to Y$ be defined by $S(x,y) = (x+\alpha, y+\beta)$, where $1,\alpha,\beta \in \mathbb{R}$ are rationally independent. So (Y,S) is equicontinuous. There exists a minimal system $(X = \mathbb{T}^2, T)$ and a factor map π from (X,T) to (Y,S) satisfying:

(1) π is of form $\pi(x,y) = (x, \varphi(x,y))$;
(2) there is some point $z_0 = (x_0,y_0) \in Y$ such that $\pi^{-1}(\{(x,y)\})$ is a singleton except when $(x,y) = S^n(x_0,y_0)$ for some $n \in \mathbb{Z}$, in which case $\pi^{-1}(\{(x,y)\})$ is an interval in $\{x\} \times \mathbb{T}$;
(3) π is asymptotic.

Now we show (X,T) is null. Let $I_n = \pi^{-1}(\{S^n(x_0,y_0)\}) \subseteq T^n(\{x_0\} \times \mathbb{T})$ for any $n \in \mathbb{Z}$. Let $\mathcal{D}$ be the decomposition of $\mathbb{T}^2$ made of intervals I_n and the individual points from the rest of the torus, i.e. $\mathcal{D} = \{\pi^{-1}(\{z\}) : z \in Y\}$. So Y is the quotient space $X/\mathcal{D}$. From here we use in Y the metric induced in the quotient space.

Fix an arbitrary sequence $A = \{t_1 < t_2 < \ldots\} \subseteq \mathbb{Z}_+$. Since $h_A(Y, S) = 0$, one has

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(S, A, \epsilon, n) = 0.$$

For any fixed $\epsilon > 0$ we are going to estimate $Span(T, A, \epsilon, n)$. Since π is asymptotic, there are finitely many intervals $J_1, \ldots, J_k$ in $\{I_n : n \in \mathbb{Z}\}$ with length greater than $\epsilon/2$. Let W' be a $(S, A, \epsilon/2, n)$-span of Y. One can assume that for any $z \in W'$, $\pi^{-1}(\{z\})$ is a singleton of X. Let $W = \pi^{-1}(W') \subseteq X$.

Consider a point $z \in X$ whose (T, A, n)-orbit $\{f^{t_i} z : 1 \le i \le n\}$ lies in $X \setminus \bigcup_{i=1}^k J_i$. If $z \in X \setminus \bigcup_{n\in\mathbb{Z}} I_n$, then since $\pi(z)$ is (S, A, ϵ, n)-spanned by W', z is (T, A, ϵ, n)-spanned by W. If $z \in I_i$ for some $i \in \mathbb{Z}$. Since $\pi(z) \in Y$ is $(S, A, \epsilon/2, n)$-spanned by $\pi(z') \in W'$ for some $z' \in X$. By the definition of quotient topology, any point in I_i is (T, A, ϵ, n)-spanned by $z' \in W$.

Now it remains to consider the points whose (T, A, ϵ, n)-orbit meet $\bigcup_{i=1}^k J_i$. Fix an $N \in \mathbb{N}$ with $1/N < \epsilon/2$. For any $1 \le i \le k$ one cuts J_i into N segments and each segment shorter than $\epsilon/2$. Let $I(t_i, J_j) = \{z \in X : T^{t_i}(z) \in J_j\}$, where $1 \le i \le n, 1 \le j \le k$. By the construction of X, $I(t_i, J_j) \in \{I_n : n \in \mathbb{Z}\}$ and each element in its (T, A, n)-orbit $\{T^{t_1} I(t_i, J_j), \ldots, T^{t_n} I(t_i, J_j)\}$ is in $\{I_n : n \in \mathbb{Z}\}$. Hence at most k of them have length greater than $\epsilon/2$. Observe one has cut every J_i, $1 \le i \le k$, into N segments with length less than $\epsilon/2$ and that one looks the interval I_i with diameter less than $\epsilon/2$ itself as one segment. Thus each point $z \in I(t_i, J_j)$ can be coded by the sequence $(S_1(z), S_2(z), \ldots, S_n(z))$, where $S_l(z)$ is the segment containing $T^{t_l} z$, $1 \le l \le n$. One has at most N^k different codes and all points with the same code can be $(T, A, \epsilon/2, n)$-spanned by one point.

To sum up, we have $Span(T, A, \epsilon, n) \le Span(S, A, \epsilon/2, n) + n \cdot k \cdot N^k$. So

$$\begin{aligned} h_A(X, T) &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(T, A, \epsilon, n) \\ &= \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(S, A, \epsilon/2, n) = 0. \end{aligned}$$

Thus $h_\infty(X, T) = 0$ and (X, T) is null.

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