THE STRUCTURE OF GRAPH MAPS WITHOUT PERIODIC POINTS

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Abstract. In this paper we show that any graph map without periodic points has only one minimal set. We describe a class of graph maps without periodic points. Our main result is to give a structure theorem of graph maps without periodic points, which states that any graph map without periodic points must be topologically conjugate to one of the described class. In addition, we give some applications of the structure theorem.

1. Introduction

A *topological dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and f is a continuous map from X to itself. A *semi-conjugation* or *factor map* $\pi : (X, f) \longrightarrow (Y, g)$ is a continuous onto map from X to Y such that $g \circ \pi = \pi \circ f$; in this situation (X, f) is said to be an *extension* of (Y, g) and (Y, g) be the *factor* of (X, f). If in addition π is a homeomorphism, then we say π is a *conjugation* and (X, f) is *conjugate* to (Y, g). For $x \in X$, $\{x, f(x), f^2(x), \ldots\}$ is called the *(positive) orbit* of x and is denoted by $Orb_+(x, f)$ (We denote $\{f^n x : n \in \mathbb{Z}\}$ by $Orb(x, f)$ when f is a homeomorphism). x is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$. Let $P(f)$ denote the set of periodic points of f. A subset M of X is *minimal* if M is nonempty, closed and invariant (i.e. $f(M) \subseteq M$) and M has no proper subset with these three properties. Note that a nonempty closed set $M \subseteq X$ is minimal if and only if the orbit of every point from M is dense in M. A dynamical system (X, f) is called *minimal* if the set X is minimal. In such a case we also say that the map f itself is minimal. A system is *transitive* if there exists some dense orbit.

If the space is an interval I, then for any continuous map $f : I \to I$ there always exist periodic points, i.e. $P(f) \neq \emptyset$. And in this case, the set $P(f)$ play a great important role in the study of dynamical properties of (I, f) [1, 10, 23]. But for another one-dimensional manifold, the circle, things become a little different. A continuous map on circle can have no periodic point, for example the irrational rotation on the unit circle. Homeomorphisms of the circle were first considered by Poincaré who used them to obtain qualitative results for a class of differential equations on torus. He classified those which have a dense orbit by showing that

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they are topologically equivalent to a rotation through an angle incommensurable with π. However, Denjoy showed that there exist homeomorphisms of the circle without periodic points and without dense orbits. This established the existence of a class of homeomorphisms of the circle without periodic points which are not conjugate to irrational rotations.

The classical theorem about homeomorphisms of the circle without periodic points is as follows. Let (S^1, f) be a homeomorphism without periodic points, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. If $\overline{Orb(x, f)} = S^1$ for some $x \in S^1$, then f is conjugate to an irrational rotation; if $\overline{Orb(x, f)} \neq S^1$ for some $x \in S^1$, then there exists a Cantor set C in S^1 which is the only minimal set of (S^1, f) and $\overline{Orb(x, f)} \setminus Orb(x, f) = C$ for all $x \in S^1 \setminus C$. For a proof we refer to [16, 21, 23]. And in [22] the author presented a classification scheme for all homeomorphisms of the circle without periodic points. In [4] the authors studied the continuous map on the circle without periodic points. They showed that if a continuous map on the circle has no periodic points, then f is semi-conjugate to an irrational rotation. This yields a complete description of all continuous mapping of the circle without periodic points.

As for general graphs, thanks to their one-dimensional character, dynamical systems on graphs have some properties which are similar to those of interval and circle maps. For example, a transitive graph map with at least one periodic point has a positive topological entropy and a dense set of periodic points [1, 2, 6]. This was first proved in [6], but unfortunately this clever paper is only available in Russian. In the same paper the author also studied the transitive graph map without periodic points and he showed a graph map without periodic points is either an irrational rotation of the circle or is monotonically (i.e. the preimage of every point is connected) and non-trivially semi-conjugate to an irrational rotation on the circle [6, 7, 8, 9].

In this paper we give a complete description of all continuous mapping of the graph without periodic points. Firstly we will study the relation between periodic points and minimal sets of a graph map. We show that if a graph map f has more than one minimal sets, then it must have periodic points. Consequently, if a graph map has no periodic point, then it has a unique minimal set. We will show that this unique minimal subsystem is either conjugate to an irrational rotation on circle or a Denjoy system (see Definition 3.7). Moreover, we will construct a class of graph maps without periodic points and show that any graph map without periodic points is conjugate to one of them. Hence we get the structure theorem of graph maps without periodic points and show the following conditions are equivalent: (i) f has no periodic point; (ii) f is semi-conjugate to an irrational rotation on the unit circle S^1; (iii) f has a unique minimal subset and f is semi-conjugate to an irrational rotation on the unit circle S^1; (iv) f is conjugate to an order n extension of an irrational rotation on unit circle S^1 for some $n \in \mathbb{N}$ (see Definition 3.4). Since we have made the structure of the continuous graph map without periodic points clear, we can use it to get lots of dynamical properties. For example, for the continuous graph map without periodic points, it is not chaotic in the sense of Li-Yorke and the topological entropy is zero. Moreover we can show it is a null system, i.e. its sequence entropy is zero for any sequence of $\mathbb{Z}_+$.

Now we introduce some definitions about graph. By a *graph* we mean a connected compact one-dimensional polyhedron in $\mathbb{R}^3$. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. Then a graph G is a continuum (i.e. a nonempty, compact, connected metric space) which can be written as the union of finitely many arcs and any two of which are either disjoint or intersect only in one endpoint. Each of these arcs is called an *edge* of the graph, and its end is called a *vertex*. Since we assume G is a polyhedron in $\mathbb{R}^3$, there are at least three edges in any circle of G. For a given graph G, a *subgraph* of G is a subset of G which is a graph itself. The *valence* of a vertex x is the number of edges that are incident on x, and if the number is n then we write $val(x) = n$. A vertex of G of valence 1 is also called an *end* of G, and a vertex x with $val(x) \geq 3$ is said a *branching point* of G. The set of ends (vertices, branching points and edges) of G will be denoted by $End(G)$ ($V(G)$, $Br(G)$ and $E(G)$ respectively). A *tree* is a graph without any subset which is homeomorphic to the unit circle. A *star* is either a tree having only one branching point or an arc. Let d be the metric on G such that for any two points $a, b \in G$, $d(a, b)$ is the minimal length of arcs in G whose endpoints are a and b. For any $a, b \in G$ with $a \neq b$, let $[a, b]$ be the arc in G whose endpoints are a and b and whose length is $d(a, b)$, if there exists a unique such arc, and define $(a, b) = [a, b] \setminus \{a, b\}$, $[a, b) = (b, a] = [a, b] \setminus \{b\}$. Hence if J is a edge of G and $a, b \in J$, then $[a, b]$ is the subarc of J whose endpoints are a and b.

Finally we give some notations used in this paper. We use $\mathbb{Z}$ ($\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ respectively) to denote the set of integers(the real numbers, the complex numbers and the natural numbers respectively) and $\mathbb{Z}_+$ the non-negative integers. For a metric space (X, d), the interior, closure and boundary of a subset $A \subseteq X$ is denoted by $int(A)$, $\overline{A}$ and ∂A respectively. Let $x \in X$, $Y \subseteq X$ and $\varepsilon > 0$. One writes $B(x, \varepsilon)$ for the ε-ball $\{x' \in X : d(x, x') < \varepsilon\}$ and $B(Y, \varepsilon) = \{x \in G : d(x, Y) < \varepsilon\}$.

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2. The graph map with at least two minimal subsets

Let G be a graph. A continuous map from G to itself is called a *graph map*, and the collection of all graph maps is denoted by $C(G)$. The main result of this section is that the graph map with at least two minimal subsets must have periodic point. To prove it, one needs the notation of the inverse orbit.

Let (X, f) be a topological dynamical system and $y_0 \in X$. If there is a sequence $(y_0, y_1, y_2, \ldots)$ such that $f(y_i) = y_{i-1}$ for any $i \geq 1$, then we call this sequence an *inverse orbit of y_0 with respect to f*. A point $x \in X$ is said to be an *α- limit point of the inverse orbit* $(y_0, y_1, y_2, \ldots)$ if there are $0 \leq n_1 < n_2 < \ldots$ such that $x = \lim_{i\to\infty} y_{n_i}$. Let Y be the set of all α- limit points of the inverse orbit $(y_0, y_1, y_2, \ldots)$ and call it *α- limit set of the inverse orbit* $(y_0, y_1, y_2, \ldots)$. It is easy to check that Y is a nonempty closed f- invariant subset of X.

Theorem 2.1. *Let G be a graph and $f \in C(G)$. If f has at least two minimal subsets, then f has periodic points.*

Proof. Assume that the theorem is not true. Then f has at least two minimal subsets but has no periodic points. First one has the following claim:

Claim. f has infinitely many minimal subsets.

Proof of the Claim. Assume that f has only finitely many minimal subsets S_1, S_2, $\ldots$, S_n, $n \geq 2$. Let $M_0 = \bigcup_{j=1}^{n} S_j$. Then M_0 is a compact subset of G. Let $\mathcal{A} = \mathcal{A}(M_0, S_1) = \{A : A$ is an arc of G, one of the ends of A is in S_1, the other is in $M_0 \setminus S_1$ and $int(A) \cap M_0 = \emptyset\}$.

Since there are only finitely many minimal subsets, it is easy to deduce that $\mathcal{A}$ is finite. For any arc $A' \in \mathcal{A}$, there is some arc $A'' \in \mathcal{A}$ such that $f(A') \supseteq A''$. Since $\mathcal{A}$ is finite, there exists some $A \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $f^k(A) \supseteq A$.

If f has no periodic point, then it is easy to see that there is some arc $A_0 \subseteq int(A)$ with $f^k(A_0) \supseteq A$. Fix any point $y_0 \in A_0$, then one can choose a sequence $(y_0, y_1, y_2, \ldots)$ from A_0 such that $f^k(y_i) = y_{i-1}$ for any $i \geq 1$, i.e. the inverse orbit of y_0 with respect to f^k. Let Y be the α- limit set of the inverse orbit $(y_0, y_1, y_2, \ldots)$. Then $Y \subseteq A_0$ and it is a nonempty closed f^k- invariant subset contained in A_0 and there is a minimal subset S_0 of f^k with

$$S_0 \subseteq Y \subseteq A_0 \subseteq int(A) \subseteq G \setminus M_0.$$

Let $S = S_0 \cup f(S_0) \cup f^2(S_0) \cup \ldots \cup f^{k-1}(S_0)$. Then S is a minimal subset of f and $S \cap M_0 = \emptyset$. But this contradicts with the fact that $S_1, S_2, \ldots, S_n$ are the all minimal sets of f. So the proof of the claim is completed.

For any two minimal sets S, S' of f, one says S and S' are *homotypic* if $(S \cup S') \cap V(G) = \emptyset$, and $S \cap E \neq \emptyset$ if and only if $S' \cap E \neq \emptyset$ for any edge E of G. Let m be the cardinality of $E(G)$ and $n = 2m + 1$. If f has no periodic point, the by Claim there exist n homotypic minimal sets $S_1, S_2, \ldots, S_n$. Again we set $M_0 = \bigcup_{i=1}^{n} S_i$. By the choice of n it is easy to show there is some $j \in \{1, 2, \ldots, n\}$ such that for any edge $E = [u, v]$ which intersects M_0, we have

$$d(S_j \cap E, u) > d((M_0 \setminus S_j) \cap E, u), \tag{1}$$

$$d(S_j \cap E, v) > d((M_0 \setminus S_j) \cap E, v). \tag{2}$$

Without loss of generality, we assume that $j = 1$. And let $\mathcal{A} = \mathcal{A}(M_0, S_1)$ as defined in the proof of Claim. For any $A' \in \mathcal{A}$, by (1) and (2) there exists a unique edge E such that $A' \subseteq int(E)$, and exists $A'' \in \mathcal{A}$ and a sub-arc A'_0 of A' with $f(A'_0) = A''$. As $\mathcal{A}$ is finite, there exists an arc $A \in \mathcal{A}$, $k \in \mathbb{N}$ and sub-arc A_0 of A such that $f^k(A_0) = A$. By this fact one can deduce that there is some point $p \in A_0$ such that $f^k(p) = p$. This contradicts with the assumption. Hence the proof of Theorem 2.1 is completed. □

Let (X, f) be a topological system. A minimal set M of f is *totally minimal* if it is a minimal set of f^n for each $n \in \mathbb{N}$. By theorem 2.1, one has

Corollary 2.2. *Let G be a graph and $f \in C(G)$. If f has no periodic point, then f has only one minimal set and this minimal set is totally minimal.*

Proof. Since f has no periodic point, f^n has no periodic point for any $n \in \mathbb{N}$. Thus, by Theorem 2.1, f^n has only one minimal set, and it is easy to see that this minimal set must be the unique minimal set of f. □

3. A class of graph maps without periodic points

In this section we construct a class of graph maps without periodic points. And in the next section we will show any graph map without periodic points is conjugate to one of them. This reveals the structure of graph maps without periodic points.

Let $I = [0, 1]$ be an interval and $C = [0, 1)$. For any $r, s \in C$ with $r \leq s$, we define

$$d_C(r, s) = min\{s - r, r + 1 - s\}.$$

Then (C, d_C) is a metric space and is isometric to the circle $\{e^{2\pi i\theta}/(2\pi) : \theta \in \mathbb{R}\}$ in the complex plane $\mathbb{C}$.

Definition 3.1. For any irrational number $r \in I$, define $h_r : C \to C$ by, for any $s \in C$,

$$\begin{cases} h_r(s) = r + s, & if\ r + s < 1; \\ h_r(s) = r + s - 1, & if\ r + s \geq 1. \end{cases}$$

The map h_r is called an *irrational rotation* of C, and r is called the *rotation number of* h_r. (see, for example, [23].)

Let T be an countable subset of $[0, 1)$. Let $\lambda : T \to (0, 1]$ be a map such that

$$l(\lambda) \equiv \sum_{t\in T} \lambda(t) < \infty.$$

For convenience, we denote $\lambda(t)$ by λ_t and for any $t \in C \setminus T$ set $\lambda_t = 0$. For any $t \in I$ let

$$J_t = \{t\} \times [0, \lambda_t] \subseteq C \times I.$$

And let

$$C_\lambda = \bigcup_{t\in C} J_t.$$

For any $x, y \in C_\lambda$, define

$$\rho_\lambda(x, y) = \min\{d_\lambda(x, y), 1 + l(\lambda) - d_\lambda(x, y)\}.$$

where d_λ is defined as follows: for any $(r, s), (r', s') \in J_\lambda$ with $r < r'$, or $r = r'$ and $s \leq s'$,

$$d_\lambda((r, s), (r', s')) = r' - r + \sum_{t\in T, r\leq t<r'} \lambda_t + s' - s.$$

It is easy to check that $(C_\lambda, \rho_\lambda)$ is a metric space and it is isometric to the circle in the complex plane $\mathbb{C}$ with radius $(1 + l(\lambda))/(2\pi)$. If there is no room for confusion, we denote the metric space (C, d_C) and $(C_\lambda, \rho_\lambda)$ by C and C_λ. *Intuitively C_λ is nothing but a circle resulting from replacing the point $t \in T$ in C by an arc J_t.*

Now we extend the irrational rotation $h_r : C \to C$ to a graph map without periodic points. First note that though $C = [0, 1)$ is a subset of $I = [0, 1]$, the metric d_C is different from the metric inherited from I. As usual, $\mathbb{R}^n$ is considered as a subspace of $\mathbb{R}^m$, where $n, m \in \mathbb{N}$ and $m > n$. That is, one regards the point $(r_1, r_2, \ldots, r_n)$ in $\mathbb{R}^n$ the same as the point $(r_1, r_2, \ldots, r_n, \underbrace{0, \ldots, 0}_{m-n})$ in $\mathbb{R}^m$.

Definition 3.2. Let T, λ, J_t and h_r be defined as above and T_0 a finite set of $C \setminus T$. Suppose that T and $T \cup T_0$ are both negative invariant set of h_r(i.e. $h_r^{-1}(T) \subseteq T$ and $h_r^{-1}(T \cup T_0) \subseteq T \cup T_0$). For any $t \in T \cup T_0$, let (G_t, d_t) be a graph in $\{t\} \times I^3 \subseteq I^4$ which satisfies the following three conditions:

(1) $J_t \subseteq G_t$;

(2) $d_t|_{J_t}$ is the same to the usual metric of J_t. That is, for any $(t, s), (t, s') \in J_t$, we have $d_t((t, s), (t, s')) = |s - s'|$.

(3) $\{t \in T \cup T_0 : G_t \setminus J_t \neq \emptyset\}$ is a finite set.

If $t \in C \setminus (T \cup T_0)$, let $G_t = \{t\}$ (as we mentioned we regard t as $t = (t, 0) \in I^2$ and $t = (t, 0, 0, 0) \in I^4$ etc.). Let $\mathbf{\Gamma} = \{G_t : t \in C\}$ and $G_{\mathbf{\Gamma}} = \bigcup_{t\in C} G_t$. Then

$$C \subseteq C_\lambda \subseteq G_{\mathbf{\Gamma}} \subseteq C \times I^3 \subseteq I^4.$$

Define the metric $d_{\mathbf{\Gamma}}$ of $G_{\mathbf{\Gamma}}$ as follows:

$$\begin{cases} d_{\mathbf{\Gamma}}(x, y) = d_t(x, y), & if\ x, y \in G_t\ for\ some\ t \in T \cup T_0; \\ d_{\mathbf{\Gamma}}(x, y) = \min\{d'(x, y), d''(x, y)\}, & if\ x \in G_s\ and\ y \in G_t\ with\ s, t \in C\ and\ s < t. \end{cases}$$

where

$$d'(x, y) = d_s(x, (s, \lambda_s)) + \rho_\lambda((s, \lambda_s), (t, 0)) + d_t((t, 0), y),$$
$$d''(x, y) = d_t(y, (t, \lambda_t)) + 1 + l(\lambda) - \rho_\lambda((s, 0), (t, \lambda_t)) + d_s((s, 0), x).$$

It is easy to check that $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ is a metric space and it is a graph. We always denote $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ by $G_{\mathbf{\Gamma}}$ for short and call $G_{\mathbf{\Gamma}}$ the *order 1 extension of the circle C*.

Remark: 1. Intuitively $G_{\mathbf{\Gamma}}$ is constructed as follows. Let $T_1 \subseteq T$ and $T_0 \subseteq C \setminus T$ be finite sets. And let $\{G_t\}_{t\in T_0\cup T_1}$ be graphs. In the circle C_λ we replace J_t by the graph $G_t(\supseteq J_t)$ for any $t \in T_0 \cup T_1$ and in addition we require that $G_t \cap C = \{(t, 0), (t, \lambda_t)\}, \forall t \in T_1$; $G_t \cap C = \{t\}, \forall t \in T_0$. Then we get the graph $G_{\mathbf{\Gamma}}$ and C_λ is the subgraph of it.

2. When $T_0 = \emptyset$ and $G_t = J_t, \forall t \in T$, we have $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}}) = (C_\lambda, \rho_\lambda)$; and when $T_0 = T = \emptyset$, we have $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}}) = (C, d_C)$. Hence we also can regard C_λ and C as the order 1 extension of the circle C.

3. Note that though C is a subset of $G_{\mathbf{\Gamma}}$, $d_{\mathbf{\Gamma}}|_C$ is different from d_C in general. When we restrict $d_{\mathbf{\Gamma}}$ on the subset C_λ of $G_{\mathbf{\Gamma}}$, it coincides with ρ_λ.

Definition 3.3. Let $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ be the graph in Definition 3.2 and $h_r : C \to C$ be an irrational rotation. Let $\varphi : G_{\mathbf{\Gamma}} \to G_{\mathbf{\Gamma}}$ be a continuous map. If $\varphi|_C = h_r$, $\varphi(t, \lambda_t) = (h_r(t), \lambda_{h_r(t)})$ and $\varphi(G_t) = G_{h_r(t)}$ for any $t \in C$, then we say φ is an *order 1 extension of h_r*.

Definition 3.4. Let $(G_1, d_1), \ldots, (G_N, d_N)$ be a sequence of graphs with $N \in \mathbb{N}$ and $f_n \in C(G_n), n = 1, \ldots, N$. If the following three conditions hold:

(1) (G_1, d_1) is the order 1 extension of circle C and f_1 is the order 1 extension of h_r, i.e. there exists some $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ and $\varphi \in C(G_{\mathbf{\Gamma}})$ as in Definition 3.3 such that $(G_1, d_1) = (G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ and $f_1 = \varphi$;
(2) for any $n \in \{1, \dots, N-1\}$, (G_n, d_n) is a proper sub-graph of (G_{n+1}, d_{n+1}), i.e. $G_n \subsetneqq G_{n+1}$ and $d_{n+1}|_{G_n} = d_n$;
(3) for any $n \in \{1, \dots, N-1\}$, $f_{n+1}(G_{n+1}) \subseteq G_n$ and $f_{n+1}|_{G_n} = f_n$,

then, for $n \in \{1, \dots, N\}$, we say that (G_n, d_n) is an order n extension of (C, d_c), and f_n is an *order n extension of h_r*, and r is called the *rotation number of f_n*.

$$\begin{array}{ccccccccccc}
(G_1,d_1) & \longleftarrow & (G_2,d_2) & \longleftarrow & \dots & \longleftarrow & (G_n,d_n) & \longleftarrow & \dots & \longleftarrow & (G_N,d_N) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow & & \downarrow{\scriptstyle f_N} \\
(G_1,d_1) & \longleftarrow & (G_2,d_2) & \longleftarrow & \dots & \longleftarrow & (G_n,d_n) & \longleftarrow & \dots & \longleftarrow & (G_N,d_N)
\end{array}$$

Now we have finished our construction. It is obvious that any order n extension of an irrational rotation has no periodic points for any $n \in \mathbb{N}$. The following proposition is easy to be verified by the definition:

Proposition 3.5. *Let $n \in \mathbb{N}$, G_n be a graph and $f_n \in C(G_n)$ be an order n extension of an irrational rotation $h_r \in C(C)$. Then f_n is semi-conjugate to h_r.*

Proof. Let $(G_1, d_1), (G_2, d_2), \dots, (G_N, d_N)$ and $f_n \in C(G_n), n = 1, 2, \dots, N$ be the sequences in Definition 3.4. First we define $\pi_1 : (G_{\mathbf{\Gamma}}, \varphi) \to (C, h_r)$ by $\pi_1(G_t) = t, \forall t \in C$. Then it is easy to verify π_1 is continuous surjective map and $\pi_1 \varphi = h_r \pi_1$, i.e. $f_1 = \varphi$ is semi-conjugate to h_r.

Now assume the semi-conjugation $\pi_i : (G_i, f_i) \to (C, h_r)$ is well defined for $i \leq n-1$. Then we define $\pi_n : (G_n, f_n) \to (C, h_r)$ by $\pi_n = h_r^{-1} \pi_{n-1} f_n$. Then

$$\pi_n f_n = h_r^{-1} \pi_{n-1} f_n f_n = h_r^{-1} \pi_{n-1} f_{n-1} f_n = h_r^{-1} h_r \pi_{n-1} f_n = h_r h_r^{-1} \pi_{n-1} f_n = h_r \pi_n.$$

That is, f_n is semi-conjugate to h_r. Thus the proof is completed. □

A point $x \in G$ is *non-wandering* if for every neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$. The set of non-wandering points of f is denoted by $\Omega(f)$. Any point not in $\Omega(f)$ is called a *wandering point*.

Proposition 3.6. *Let $n \in \mathbb{N}$, G_n be a graph and $f_n \in C(G_n)$ be an order n extension of an irrational rotation $h_r \in C(C)$. Then f_n has a unique minimal set $S_T = C \cup \{(t, \lambda_t) : t \in T\}$ and any other point in G_n is the wandering point of f_n.*

Proof. Assume Proposition 3.6 holds for $n = n_0 \in \mathbb{N}$, then by Definition 3.4 it still holds for $n = n_0 + 1$. Hence it suffices to show the case $n = 1$. By Definition 3.2 and Definition 3.3, we have $G_1 = G_{\mathbf{\Gamma}}$ and $f_1 = \varphi$.

Let $S_T = C \cup \{(t, \lambda_t) : t \in T\}$. Obviously, $S_T = \{(t, 0), (t, \lambda_t) : t \in C\}$. That is, when $t \in T$, $(t, 0) = t$ and (t, λ_t) are the two endpoints of the interval J_t, and when $t \in C \setminus T$, $(t, \lambda_t) = (t, 0) = t$.

By the construction of $(G_{\mathbf{\Gamma}}, d_{\mathbf{\Gamma}})$ as in Definition 3.2, for any $s \in C$ and $\varepsilon > 0$, there is small enough $\delta > 0$ such that

$$\max\{d_{\mathbf{\Gamma}}(x, (s, \lambda_s)) : x \in \bigcup\{G_t : s < t < s + \delta\}\} < \varepsilon,$$

and

$$\max\{d_{\mathbf{\Gamma}}(y, (s, 0)) : y \in \bigcup\{G_t : s - \delta < t < s\}\} < \varepsilon.$$

(When $s = 0$, then replace $\{G_t : s - \delta < t < s\}$ by $\{G_t : 1 - \delta < t < 1\}$.) For any $s' \in C$, since h_r is minimal, there are some $m, k \in \mathbb{N}$ such that $s < h_r^m(s') < s + \delta$ and $s - \delta < h_r^k(s') < s$ (when $s = 0$, we replace the latter by $1 - \delta < h_r^k(s') < 1$). Note that $\varphi^n(G_{s'}) = G_{h_r^n(s')}, n = m, k$. We have $\{(s, 0), (s, \lambda_s)\} \subseteq \omega(z, \varphi)$ for any $s, s' \in C$ and any $z \in G_{s'}$. So S_T must be contained in some minimal set of φ.

On the other hand, we have $G_{\mathbf{\Gamma}} \setminus S_T = \bigcup\{G_t \setminus \{(t, 0), (t, \lambda_t)\} : t \in T \cup T_0\}$. For any $t \in T \cup T_0$ and $x \in G_t \setminus \{(t, 0), (t, \lambda_t)\}$, by the definition of φ and $d_{\mathbf{\Gamma}}$ we have

$$d_{\mathbf{\Gamma}}(x, \bigcup\{\varphi^n(G_t) : n \in \mathbb{N}\}) = \min\{d(x, (t, 0)), d(x, (t, \lambda_t))\} > 0.$$

Hence every point in $G_{\mathbf{\Gamma}} \setminus S_T$ is the wandering point of φ, and S_T is the only minimal set of φ. □

If $T = \emptyset$, then $S_T = C$ is a circle. If $T \neq \emptyset$, then $S_T = C \cup \{(t, \lambda_t) : t \in T\}$ is a Cantor set of C_λ. In this case one can define a semi-conjugation $\psi : (S_T, \varphi) \to (C, h_r)$ with

$$\psi(x) = \begin{cases} x, & x \in C \setminus T; \\ t, & x \in \{(t, 0), (t, \lambda_t)\} \text{ for } t \in T. \end{cases}$$

Then (S_T, φ) is an almost one-to-one extension of (C, h_r).

Definition 3.7. If $T \neq \emptyset$, then we call the minimal system (S_T, φ) *Denjoy system.*

We end this section with some easy observations.

Proposition 3.8. *Let $n \in \mathbb{N}$, G_n be a graph and $f_n \in C(G_n)$ be an order n extension of an irrational rotation $h_r \in C(C)$. If f_n is surjective, then $n = 1$. Moreover, f_n is a homeomorphism if and only if $G_n = C_\lambda$ and (C_λ, φ) is a homeomorphic system.*

Proposition 3.9. *Let $n \in \mathbb{N}$, G_n be a graph and $f_n \in C(G_n)$ be an order n extension of an irrational rotation $h_r \in C(C)$. Then f_n is transitive if and only if (G_n, f_n) is an irrational rotation (C, h_r).*

4. THE STRUCTURE OF GRAPH MAPS WITHOUT PERIODIC POINTS

In this section we give the structure of the graph maps without periodic points. We show that any graph map without periodic points is conjugate to some order n extension of an irrational rotation $h_r \in C(S^1)$ defined in the last section (Definition 3.4). Firstly one has

Theorem 4.1. *Let G be a graph and $f : G \to G$ a continuous surjective map. If f has no periodic point, then there exists an irrational number $r \in I$ such that f is conjugate to some order 1 extension of the irrational rotation $h_r : C \to C$.*

Proof. The proof is divided into several claims. Firstly by Corollary 2.2 and Theorem 1 in [5], one has:

Claim 1. f has a unique minimal set S which is totally minimal. And S is either a Cantor set or a circle. Especially, there is no isolated point in S.

It is well known that a minimal map is semi-open, i.e. the image of any non-empty open set has non-empty interior (see, for example, [12, 18]). Hence for the minimal map $f|_S : S \to S$, one has:

Claim 2. For any $x \in S$ and $\varepsilon > 0$, there is no isolated point in $f(B(x, \varepsilon) \cap S)$.

For any $x \in S$, let $n = val(x)$ and

$$c_x = d(x, V(G) \setminus \{x\})/3.$$

Then $\overline{B(x, c_x)}$ is a n- star, and we call it a *standard closed neighborhood of x*. Denote all ends of $\overline{B(x, c_x)}$ by $\{y_1, y_2, \ldots, y_n\}$. For $i \in \{1, 2, \ldots, n\}$, the arc $[x, y_i]$ is said to be a *standard neighborhood branch of x*. For any $\varepsilon \in (0, c_x]$, we call $[x, y_i] \cap \overline{B(x, \varepsilon)}$ an *ε- neighborhood branch of x*. If for any $\varepsilon > 0$ we have $(x, y_i] \cap B(x, \varepsilon) \cap S \neq \emptyset$, then $[x, y_i]$ is said to be a *standard effective branch of x (with respect to S)*, otherwise it is said to be a *standard ineffective branch of x (with respect to S)*. Let $\mu(x)$ be the cardinality of the standard effective branches of x. By Claim 1, one has

$$1 \leq \mu(x) \leq val(x), \quad \forall x \in S. \tag{3}$$

For any subset Y of G, if $Y \cap S = \emptyset$, then let $\mu(Y) = 0$. And if $Y \cap S \neq \emptyset$ is finite, then let

$$\mu(Y) = \sum_{x \in Y \cap S} \mu(x), \tag{4}$$

and we call $\mu(Y)$ the *number of effective branches which Y adjoins.*

Claim 3. Let A be an arc of G with $int(A) \cap V(G) = \emptyset$. If there is some closed arc A_1 of G with $f(A_1) \supseteq A$, then $f^{-1}(int(A)) \cap (S \setminus A_1) = \emptyset$.

Proof of Claim 3. Suppose that Claim 3 does not hold, then there exists $w \in S \setminus A_1$ such that $f(w) \in int(A)$. Hence there is some $\varepsilon > 0$ such that $B(w, \varepsilon) \cap A_1 = \emptyset$ and $f(B(w, \varepsilon)) \subseteq int(A)$. Since f is surjective, we can choose an inverse orbit $y_0, y_1, y_2, \ldots$ such that $y_0 \in A$ and we choose y_n in A_1 when $y_{n-1} \in A$, $n \geq 1$. Hence we get an inverse orbit $(y_0, y_1, y_2, \ldots)$ with respect to f which is disjoint from $B(w, \varepsilon)$. As $B(w, \varepsilon)$ is open, the α- limit set Q of $(y_0, y_1, y_2, \ldots)$ is also disjoint from $B(w, \varepsilon)$. Let S' be some minimal set contained in Q. Since $w \notin S'$, we have $S' \neq S$. This contradicts with that f has only one minimal subset. The proof of Claim 3 is completed.

Claim 4. Let $x \in S$, $x' = f(x)$, and $[x, y]$ be a standard effective branch with respect to S. Let

$$r_0 = \sup\{r \in (0, c_x] : f(B(x, r)) \subseteq B(f(x), c_{x'})\}, \tag{5}$$

and $w \in (x, y]$ with $d(w, x) = r_0$. Then there exists a unique standard effective branch $[x', y']$ of x' such that $f([x, w]) \subseteq [x', y']$.

Proof of Claim 4. By Claim 2, there exist a sequence $w_1, w_2, \ldots$ in $(x, w] \cap S$ and a standard effective branch $[x', y']$ of x' such that $\lim_{n\to\infty} w_n = x$ and $f(\{w_1, w_2, \ldots\}) \subseteq (x', y']$. If Claim 4 doesn't hold, then there is another standard effective branch $[x', y'']$ of x' and some $x_1 \in (x, w]$ such that $f(x_1) \in (x', y'']$. Let

$$A = [f(x_1), x'] \cup [x', f(w_1)], \text{ and } A_1 = [x_1, w_1].$$

Then A, A_1 are arcs and $f(A_1) \supseteq A$. Take $n \in \mathbb{N}$ such that $w_n \in [x, w_1) \cap [x, x_1)$ and $f(w_n) \in [x', f(w_1))$. Then $w_n \in S \setminus A_1$ and $f(w_n) \in int(A)$. But this contradicts Claim 3. Hence the proof of Claim 4 is completed.

A subset Y of G is called *S- finite* if $Y \cap S$ is a finite set.
Claim 5. If Y is an S-finite connected closed subset, then $f(Y)$ is an S-finite connected closed subset too.
Proof of Claim 5. Let $K = f(Y)$. As Y is connected and closed, so is K. If K is not S- finite, then there is an arc $A_1 = [x, y]$ in Y and an arc $A = [x', y']$ in K such that $x' = f(x) \in S, A \subseteq f(A_1)$, $(x, y] \cap S = \emptyset$ and $int(A) \cap S \neq \emptyset$. This contradicts with Claim 3. Thus K is S- finite.

By Claim 4 and Claim 5, one has
Claim 6. Let $x \in S$, $x' = f(x)$, and r_0 be defined by (5). If $[x, y_1]$ is a standard effective branch of x, $[x, y_2]$ is a standard ineffective branch of x, $[x', y_3]$ is a standard effective branch of x' and $[x', y_4]$ is a standard ineffective branch of x', then

$$f([x, y_1] \cap B(x, r_0)) \cap (x', y_4] = \emptyset,$$

and

$$f([x, y_2] \cap B(x, r_0)) \cap (x', y_3] = \emptyset.$$

Claim 6 implies that when the neighborhood is small enough, f can not map the effective branch of x to the ineffective branch of $f(x)$, and can not map the ineffective branch of x to the effective branch of $f(x)$ too.

By Claim 3 one has Claim 7 readily.
Claim 7. Let $x, x_1 \in S$ (x may equal x'), $[x, y]$ and $[x_1, y_1]$ be the standard effective branches of x and x_1 respectively and $[x, y] \neq [x_1, y_1]$. Assume that $f(x) = f(x_1) = x'$ and r_0, w and $[x', y']$ are defined as in Claim 4. Let

$$r_1 = \sup\{r \in (0, c_x] : f(B(x, r)) \subseteq B(f(x_1), c_{x'})\},$$

and $w_1 \in (x_1, y_1]$ with $d(w_1, x_1) = r_1$. Then $f([x_1, w_1]) \cap (x', y'] = \emptyset$.
Claim 7 implies that if the neighborhood is small enough, then f can not map the different effective branches to the same effective branch.

By (4), $\mu(Y)$ is an non-decreased function on Y, i.e. if Y and Y' are S- finite set with $Y \subseteq Y'$, then $\mu(Y) \leq \mu(Y')$. Note that $f(S) = S$, and by claims 4, 5 and 7, we have Claim 8 immediately.
Claim 8. Let Y be an S-finite connected closed subset. Then

$$\mu(f(Y)) \geq \mu(Y).$$

Especially, for any $x \in S$. we have $\mu(f(x)) \geq \mu(x)$.

Claim 9. For any $x \in S$, $1 \le \mu(x) \le 2$.
Proof of Claim 9. In (3), we have shown $\mu(x) \ge 1$. Now we prove $\mu(x) \le 2$. Assume $\mu(x) \ge 3$, then by Claim 8 for any $n \in \mathbb{N}$ one has

$$val(f^n(x)) \ge \mu(f^n(x)) \ge \mu(x) \ge 3.$$

This means that there are infinitely many branching points in graph G, a contradiction! Hence $\mu(x) \le 2$.

An S- finite connected closed set Y is *maximal* if there is no S- finite connected closed set Y_1 with $Y \subsetneq Y_1$.
Claim 10. If Y is a maximal S- finite connected closed set, then $\partial Y \subseteq S$ and for any arc $A = [x, y]$ of G with $x \in \partial Y$ and $(x, y] \cap Y = \emptyset$, one has $(x, y] \cap S \ne \emptyset$.
Proof of Claim 10. If Claim 10 does not hold, then one can find an arc $A = [x, y]$ of G such that $x \in \partial Y$, $int(A) \cap (Y \cup S) = \emptyset$ and $y \in S$. Hence $Y \cup A \supsetneq Y$ is an S-finite connected closed set, which contradicts with the maximality of Y.

Claim 11. If Y is an S- finite connected closed set, then $\mu(Y) \le 2e(G)$, where $e(G)$ is the cardinality of $E(G)$.
Proof of Claim 11. Let $K = \overline{B(Y, \varepsilon)}$, where $\varepsilon > 0$ is small enough and $B(Y, \varepsilon) = \{x \in G : d(x, Y) < \varepsilon\}$. Then $\partial K \subseteq G \setminus V(G)$. Hence $\mu(Y) \le \#(\partial K) \le 2e(G)$, where $\#T$ denote the cardinality of a set T.

The following claim follows from Claim 11.
Claim 12. The closure of any S- finite connected set and the union of two intersecting S- finite connected sets are S- finite connected sets. And any S- finite connected closed set is contained in some maximal S- finite connected closed set.

Claim 13. If Y is an S- finite connected closed set, then $f^n(Y) \cap f^k(Y) = \emptyset$ for any two integers $n > k \ge 0$.
Proof of Claim 13. Suppose that there are integers $n > k \ge 0$ such that $f^n(Y) \cap f^k(Y) \ne \emptyset$. Let W be the maximal S- finite connected closed set containing Y. Then $f^{n-k}(W) \cap W \ne \emptyset$. By Claim 5 and Claim 12, $f^{n-k}(W) \cup W$ is also an S- finite connected closed set. Hence $f^{n-k}(W) \subseteq f^{n-k}(W) \cup W = W$ and $f^{n-k}(W \cap S) \subseteq W \cap S$. Since $W \cap S$ is finite, there must be some periodic point in $W \cap S$. This contradicts with the assumption.

Claim 14. Let Y be an S- finite connected closed set with $\mu(Y) \ge 3$. Then $Y \cap Br(G) \ne \emptyset$.
Proof of Claim 14. If $Y \cap Br(G) = \emptyset$, then Y is an arc and there is some edge E of G with $Y \subseteq int(E) \cup End(G)$. Let $End(Y) = \{u, v\}$. Then we have $\mu(Y) = \mu(u) + \mu(v) \le 1 + 1 = 2$. This contradicts with the condition $\mu(Y) \ge 3$.

Claim 15. Let Y be an S- finite connected closed set, then $\mu(Y) \le 2$.
Proof of Claim 15. If $\mu(Y) \ge 3$, then by Claim 8 $\mu(f^n(Y)) \ge 3$ for any $n \in \mathbb{Z}_+$. According to Claim 13 and Claim 14 we deduce that there are infinitely many branching points located in $Y, f(Y), f^2(Y), \ldots$. This is impossible. So we must have $\mu(Y) \le 2$.

Claim 16. Let Y be a maximal S- finite connected closed set, then $\mu(Y) = 2$.

Proof of Claim 16. By Claim 10 there is some point x_0 in $Y \cap S$. As $f(S) = S$, we can choose an inverse orbit $(x_0, x_1, x_2, \ldots)$ in S. There exists some edge E of G and $i, k, n \in \mathbb{N}$ such that $\{x_j, x_k, x_n\} \subseteq E$ and $x_j < x_n < x_k$, where the order $<$ is the usual order in the interval. If $\mu(x_n) \geq 2$, then by Claim 8 we have

$$\mu(Y) \geq \mu(x_0) = \mu(f^n(x_n)) \geq \mu(x_n) \geq 2.$$

If $\mu(x_n) < 2$, then we will show $\mu(Y) \geq 2$. First by (3) we have $\mu(x_n) = 1$. Let Y_n be the maximal S-finite connected closed set containing x_n. Then Y_n is an arc contained in $[x_j, x_k]$ and let $Y_n = [u, v]$. So $x_n \in \{u, v\} = Y_n \cap S$ and $\mu(Y_n) = \mu(u) + \mu(v) = 1 + 1 = 2$. By Claim 5 we have $x_0 = f^n(x_n) \in f^n(Y_n) \subseteq Y$. By Claim 8 we have $\mu(Y) \geq \mu(Y_n) \geq 2$. Thus in both cases we have $\mu(Y) \geq 2$, and by Claim 15 $\mu(Y) = 2$.

For any $x \in S$, Let Y_x denote the maximal S-finite connected closed set containing x. Let $\mathbf{Y} = \{Y_x : x \in S\}$. By Claim 12 $\bigcup\{Y_x : x \in S\} = G$ and $Y_x \cap Y_y = \emptyset$ if $Y_x \neq Y_y$.

Claim 17. For any $Y \in \mathbf{Y}$ one has $f^{-1}(Y) \in \mathbf{Y}$ and $f(Y) \in \mathbf{Y}$.

Proof of Claim 17. Since f is surjective, for any $Y \in \mathbf{Y}$ there is $Y' \in \mathbf{Y}$such that $f(Y') \cap Y \neq \emptyset$. According to Claim 5 and Claim 12, we have $f(Y') \subseteq Y$, i.e. $f^{-1}(Y) \supseteq Y'$. If $f^{-1}(Y) \neq Y'$, then there is some $Y'' \in \mathbf{Y} \setminus \{Y'\}$ such that $f(Y'') \cap Y \neq \emptyset$. At this case we also have $f(Y'') \subseteq Y$. By Claim 7 and Claim 16, $\mu(Y) \geq \mu(Y') + \mu(Y'') = 4$. This contradicts with Claim 16. So we have $f^{-1}(Y) = Y' \in \mathbf{Y}$. The proof for $f(Y) \in \mathbf{Y}$ is similar.

Let

$$\mathbf{Y}_i = \{Y_x : x \in S, \#(Y_x \cap S) = i\},\ i = 1, 2.$$

By Claim 9 and Claim 16, $\mathbf{Y} = \mathbf{Y}_1 \cup \mathbf{Y}_2$. Let

$$\mathbf{Y}_{11} = \{Y \in \mathbf{Y}_1 : Y = Y \cap S\},$$

$$\mathbf{Y}_{12} = \{Y \in \mathbf{Y}_1 : Y \setminus S \neq \emptyset\},$$

$$\mathbf{Y}_{21} = \{Y \in \mathbf{Y}_2 : Y \text{ is an arc and } End(Y) = Y \cap S\},$$

$$\mathbf{Y}_{22} = \mathbf{Y}_2 \setminus \mathbf{Y}_{21}.$$

Then $\mathbf{Y}_i = \mathbf{Y}_{i1} \cup \mathbf{Y}_{i2}, i = 1, 2$. Note that for any $Y \in \mathbf{Y}_2$, $Y \in \mathbf{Y}_{22}$ if and only if either Y is not an arc, or Y is an arc but $(Y \setminus End(Y)) \cap S \neq \emptyset$. It is easy to show:

Claim 18. If $Y \in \mathbf{Y}_{12} \cup \mathbf{Y}_{21} \cup \mathbf{Y}_{22}$, then Y is the union of finitely many connected components of $G \setminus S$. Moreover, since $G \setminus S$ has at most countably many connected components, $\mathbf{Y}_{12} \cup \mathbf{Y}_{21} \cup \mathbf{Y}_{22}$ is countable.

By Claim 17 one has:

Claim 19. If $Y \in \mathbf{Y}_{12}$, then $f(Y) \in \mathbf{Y}_{12} \cup \mathbf{Y}_{11}$.

By Claim 9 and Claim 16 one has:

Claim 20. For any $Y \in \mathbf{Y}_{11} \cup \mathbf{Y}_{21}$, one has $Y \cap Br(G) = \emptyset$. For any $Y \in \mathbf{Y}_{12} \cup \mathbf{Y}_{22}$, one has $Y \cap Br(G) \neq \emptyset$. And if $Y = Y_x \in \mathbf{Y}_{12}$ for some $x \in S$, then $x \in Y \cap Br(G)$. Moreover, as $Br(G)$ is finite, $\mathbf{Y}_{12} \cup \mathbf{Y}_{22}$ is finite.

Let

$$Z_j = \bigcup\{Y : Y \in \mathbf{Y}_{1j} \cup \mathbf{Y}_{2j}\}, j = 1, 2.$$

Then $Z_1 \cap Z_2 = \emptyset$ and $Z_1 \cup Z_2 = G$. By Claim 20, one has:

Claim 21. *Z_1 is a one dimension manifold. Every connected component of Z_1 is an open arc contained in some connected component of $G \setminus Br(G)$, and every connected component of $G \setminus Br(G)$ contains at most one component of Z_1. $Br(G) \subseteq Z_2$, and every connected component contains at least one branching point.*

Especially, the number of connected components of Z_1 is less than $e(G)$ and the number of connected components of Z_2 is less than the number of $Br(G)$.

By Claim 21 one has

$$\overline{Z_1} \setminus Z_1 = \overline{Z_1 \cap S} \setminus (Z_1 \cap S) \subseteq \overline{S} = S.$$

Note that $\overline{Z_1} \setminus Z_1 \subseteq G \setminus Z_1 = Z_2$, one has $\overline{Z_1} \setminus Z_1 \subseteq Z_2 \cap S$. Conversely, by Claim 2 one has $S \subseteq \overline{Z_1}$. So

$$Z_2 \cap S = (G \setminus Z_1) \cap S \subseteq (G \setminus Z_1) \cap \overline{Z_1} = \overline{Z_1} \setminus Z_1.$$

Thus one gets the following claim:

Claim 22. $\overline{Z_1} \setminus Z_1 = Z_2 \cap S$.

For any $x \in S$, if $Y_x \in \mathbf{Y}_1$, then let $J_x = J(Y_x) = \{x\}$; if $Y_x \in \mathbf{Y}_{21}$, then let $J_x = J(Y_x) = Y_x$; if $Y_x \in \mathbf{Y}_{22}$ and $Y_x \cap S = \{x, x'\}$, then let $J_x = J_{x'} = J(Y_x)$ be any arc in Y_x with ends $\{x, x'\}$. So if $Y_x \subseteq Z_1$, then $J_x = Y_x$; and if $Y_x \subseteq Z_2$, then J_x is a proper subset of Y_x. Let

$$G_0 = \bigcup\{J_x : x \in S\}.$$

Then $S \subseteq G_0$ and $G_0 = Z_1 \cup Z_3 \cup Z_4$, where $Z_3 = \{x \in S : Y_x \in \mathbf{Y}_{12}\}$ and $Z_4 = \bigcup\{J_x : x \in S \text{ and } Y_x \in \mathbf{Y}_{22}\}$. That is, G_0 is obtained by connecting all the ends of every connected components of Z_1 (note that they are open arcs) using the points of Z_3 and the closed arcs of Z_4. So G_0 is the union of finitely many non-degenerate connected closed subsets of G. By Claim 9 and Claim 16, there are neither ends nor branching points in G_0. Thus G_0 is the union of finitely many disjoint circles of G. Moreover, by Claim 21 and Claim 22, one has

Claim 23. *For any $Y \in \mathbf{Y}_{12} \cup \mathbf{Y}_{22}$,*

$$Y \cap \overline{G \setminus Y} = Y \cap \overline{Z_1} = Y \cap (\overline{Z_1} \setminus Z_1) = Y \cap S.$$

By Claim 23 after one replaces every connected closed subset $Y \in \mathbf{Y}_{12} \cup \mathbf{Y}_{22}$ of G by the subset $J(Y)$ of Y, it will not destroy the connectedness of the graph G. Hence G_0 is connected. That is,

Claim 24. *G_0 is a circle containing the minimal set S of f.*

Obviously, one can define a continuous map $f_0 : G_0 \to G_0$ such that $f_0|_S = f|_S$ and for any $x \in S$, $f_0(J_x) = J_{f(x)}$. By Claim 13 and Claim 17, for any $x \in S$ and any integers $n > k \geq 0$, one has

$$f_0^n(J_x) \cap f_0^k(J_x) = J_{f^n(x)} \cap J_{f^k(x)} \subseteq Y_{f^n(x)} \cap Y_{f^k(x)} = f^n(Y_x) \cap f^k(Y_x) = \emptyset.$$

Hence f_0 has no periodic point.

Let $G_0^* = \{J_x : x \in S\}$. Then G_0^* is a partition of G_0. Let τ^* be the identification topology of G_0^* induced by the topology of G_0 (see, for example, Page 66 in [3]). If there is no confusion we will denote (G_0^*, τ^*) by G_0^*. It is easy to check that G_0^* is homeomorphic to the circle (C, d_C). Define $f_0^* : G_0^* \to G_0^*$ by $f_0^*(J_x) = J_{f(x)} (= f_0(J_x)), \forall J_x \in G_0^*$. It is easy to check (G_0^*, f_0^*) is a minimal homeomorphism. Let $r \in [0,1)$ be its rotation number, then r is an irrational number. Let $h_r : C \to C$ is the irrational rotation defined in Definition 3.1 Then f_0^* is conjugate to h_r, i.e. there exists a homeomorphism $\psi : G_0^* \to C$ such that $\psi f_0^* = h_r \psi$.

For $i, j \in \{1, 2\}$ let

$$S_{ij} = \{x \in S : Y_x \in \mathbf{Y}_{ij}\}$$

and

$$T_{ij} = \{\psi(J_x) : x \in S_{ij}\}.$$

Then T_{12} and T_{22} are finite subsets of C and T_{21} is a countable subset of C. Let

$$T = T_{21} \cup T_{22}, \text{and } \ T_0 = T_{12}.$$

Then T and $T \cup T_0$ are negative invariant subsets of h_r. Take an arbitrary function $\lambda : T \to (0, 1]$ with $\sum_{t \in T} \lambda(t) < \infty$. As before denote $\lambda(t)$ by λ_t and let $\lambda_t = 0$ when $t \in C \setminus T$. For any $t \in C$, let $J_t = \{t\} \times [0, \lambda_t]$ (when $t \in C \setminus T$, J_t is a singleton $\{(t, 0)\}$). Let $C_\lambda = \bigcup\{J_t : t \in C\}$ and the metric ρ_λ of C_λ as in Definition 3.1. Let

$$C' = \{(t, \lambda_t) \in I^2 : t \in C\}.$$

Then $C' \cup C \subseteq C_\lambda$ and

$$C' \cap C = \{(t, 0) \in I^2 : t \in T_{11} \cup T_0\} = (T_{11} \cup T_0) \times \{0\} = T_{11} \cup T_0.$$

It is obvious that there is a homeomorphism $H_0 : G_0 \to C_\lambda$ such that $H_0(S) = C \cup C'$ and for any $x \in S$ we have $H_0(J_x) = J_{\psi(J_x)}$, $H_0(J_{f(x)}) = H_0 f_0^*(J_x) = J_{\psi f_0^*(J_x)} = J_{h_r \psi(J_x)}$. And if $H_0(x) = t \in C$, then $H_0(f(x)) = h_r(t) \in C$; if $H_0(x) = (t, \lambda_t) \in C'$, then $H_0(f(x)) = (h_r(t), \lambda_{h_r(t)}) \in C'$.

For any $t \in T_0 \cup T_{22}$, let $x_t = H_0^{-1}(t)$. Choose a graph $(G_t, d_t) \subseteq \{t\} \times I^3$ which is homeomorphic to Y_{x_t} and a homeomorphism $H_{x_t} : Y_{x_t} \to G_t$ such that the conditions (1) and (2) in Definition 3.2 hold and $H_{x_t}|_{J_{x_t}} = H_0|_{J_{x_t}}$. For any $t \in T_{11} \cup T_{21}$, let $G_t = J_t$. Let $G_{\mathbf{\Gamma}} = \bigcup\{G_t : t \in C\}$ and the metric $d_{\mathbf{\Gamma}}$ be the metric defined in Definition 3.2. Define $H : G \to G_{\mathbf{\Gamma}}$ as

$$\begin{cases} H|_{G_0} = H_0; \\ H|_{Y_{x_t}} = H_{x_t}, t \in T_0 \cup T_{22}; \\ H|_{Y_x} = H|_{J_x} = H_0|_{J_x}, x \in S_{11} \cup S_{21}. \end{cases}$$

By Claim 23 H is a homeomorphism.

$$\begin{array}{ccccc} (G, f) & \longrightarrow & (G_0, f_0) & \longrightarrow & (G_0^*, f_0^*) \\ \downarrow H & & \downarrow H_0 & & \downarrow \psi \\ (G_{\mathbf{\Gamma}}, \varphi) & \longrightarrow & (C_\lambda, \varphi) & \longrightarrow & (C, h_r) \end{array}$$

Define $\varphi : G_{\mathbf{\Gamma}} \to G_{\mathbf{\Gamma}}$ by $\varphi = HfH^{-1}$. Then φ is conjugate to f and φ satisfies all conditions in Definition 3.3. That is, φ is an order 1 extension of the irrational rotation $h_r : C \to C$. Thus the proof of the theorem is completed. □

Theorem 4.2. *Let G be a graph and $f : G \to G$ a continuous map. Then f has no periodic point if and only if there exists an irrational number $r \in I$ and $n \in \mathbb{N}$ such that f is conjugate to some order n extension of the irrational rotation $h_r : C \to C$.*

Proof. By Proposition 3.5, the sufficiency is obvious. Now we show the necessary part of theorem. Assume that f has no periodic point, by Theorem 4.1 we need only consider the case when f is not surjective.

For any $i \in \mathbb{Z}_+$, let $X_i = f^i(G)$. Then X_i is a connected closed subset of G and $G = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$. Let $K_1 = \bigcap_{i=0}^{\infty} X_i$. Then K_1 is also a connected closed subset of G and $f(K_1) = K_1$. Since f has no periodic point, $X_1, X_2, \dots$ and K_1 contain at least two points and hence all are the sub-graphs of G. Let $f_1 = f|_{K_1} : K_1 \to K_1$. Then f_1 is a surjective graph map. By Theorem 4.1, there exist some irrational number $r \in C$ and an irrational rotation $h_r : C \to C$ such that f_1 is conjugate to some order 1 extension $\varphi : G_{\mathbf{\Gamma}} \to G_{\mathbf{\Gamma}}$ of h_r.

Let ∂K_1 be the boundary of K_1 in G. Then ∂K_1 is a finite subset of K_1. Since f_1 has no periodic point, there is some $m \in \mathbb{N}$ such that $f^m(\partial K_1) \subseteq K_1 \setminus \partial K_1$. By the continuity of f^m, there is some $\varepsilon > 0$ such that $f^m(B(\partial K_1, \varepsilon)) \cap \partial K_1 = \emptyset$ implies that $f^m(B(\partial K_1, \varepsilon)) \subseteq K_1$. By $K_1 = \bigcap_{i=0}^{\infty} X_i$, $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ and the compactness of G, there is some $j \in \mathbb{N}$ such that

$$f^{j+m}(G) = X_j \subseteq B(K_1, \varepsilon) = K_1 \cup B(\partial K_1, \varepsilon).$$

Hence we have $f^{j+m}(G) \subseteq f^m(K_1 \cup B(\partial K_1, \varepsilon)) = K_1$.

Set $n = \min\{i \in \mathbb{N} : f^i(G) \subseteq K_1\} + 1$. Then $n \le j + m + 1$ and $f^{n-1}(G) = K_1$. Let $K_i = f^{n-i}(G), i = 2, 3, \dots, n$. Then $K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots \subseteq K_n$, $f(K_i) = K_{i-1}$ and $K_i \setminus K_{i-1} \neq \emptyset$ for $i = 2, 3, \dots, n$. It is easy to verify $K_1, K_2, \dots, K_n$ satisfy the conditions in Definition 3.4. Hence $f|_{K_i} : K_i \to K_i$ is conjugate to some order i extension of h_r, $i = 1, 2, \dots, n$. Especially, $f = f|_{K_n}$ is conjugate to some order n extension of the irrational rotation $h_r : C \to C$. □

By Proposition 3.5, 3.6 and Theorem4.2, we have

Theorem 4.3. *Let G be a graph and $f \in C(G)$. The the following conditions are equivalent:*

(i) f has no periodic point;

(ii) f is semi-conjugate to an irrational rotation on the unit circle S^1;

(iii) f has only one minimal set and f is semi-conjugate to an irrational rotation on the unit circle S^1;

(iv) f is conjugate to a system which is the order n extension of an irrational rotation on the unit circle S^1 for some $n \in \mathbb{N}$.

Corollary 4.4. *Let G be a graph and $f : G \to G$ a continuous map with no periodic point. Then f has a unique minimal set S which is either conjugate to an irrational rotation on the unit circle or a Denjoy system. Moreover, we have $\Omega(f) = S$.*

Proof. The first part has been shown in the proof of Theorem 4.1. The latter follows from Theorem 4.2 and Proposition 3.6. □

Remark 4.5. For more discussion on the minimal sets for continuous graph maps, please see [5, 20] etc.

5. Applications

In this section we use the structure theorem built in the last section to get some dynamical properties of the graph maps without periodic points.

A topological dynamical system (X, f) is *equicontinuous* if for any $\epsilon > 0$ there is $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n x, f^n y) < \epsilon$ for every $n \in \mathbb{Z}_+$. (X, f) is *distal* if $\liminf d(f^n x, f^n y) > 0$ for any distinct $x, y \in X$.

It is well known that a homeomorphism h from a circle to itself without periodic points is equicontinuous if and only if h is transitive [19]. Therefore, by Theorem 5.2 of [19] and Theorem 4.2 of this paper one obtains the following proposition at once.

Proposition 5.1. *Let G be a graph and $f : G \to G$ a continuous map without periodic points. Then f is equicontinuous if and only if there is a circle C in G such that $f(C) = C$ and $f|_C$ is a transitive homeomorphism.*

Proposition 5.2. *Let G be a graph and $f : G \to G$ a continuous map without periodic points. Then*

(1) *f is transitive if an only if f is minimal if and only if f is conjugate to an irrational rotation on unit circle.*
(2) *f is a homeomorphism if and only if it is conjugate to a homeomorphic system (C_λ, φ).*
(3) *The following conditions are equivalent:*
   (a) *f is equicontinuous;*
   (b) *$int(\Omega(f)) \neq \emptyset$;*
   (c) *the unique minimal set of f is conjugate to an irrational rotation on the unit circle.*

Proof. (1) It follows from Proposition 3.9 and Theorem 4.2.
(2) It follows from Proposition 3.8 and Theorem 4.2.
(3) It follows from Corollary 4.4 and Theorem 4.2.

□

Corollary 5.3. *Let G be a graph and $f : G \to G$ a continuous surjective map without periodic points. Then the following conditions are equivalent:*

(1) *f is transitive;*
(2) *f is minimal;*
(3) *f is equicontinuous;*
(4) *f is distal;*
(5) *$int(\Omega(f)) \neq \emptyset$;*
(6) *f is conjugate to an irrational rotation on the unit circle.*

Proof. If f is surjective, then the equicontinuity implies distality. And every point in a distal system is minimal (see, for example, [12]). Hence the results follow from Corollary 5.2 and Theorem 4.2. □

By Theorem 4.3 one can see that the dynamical properties of graph maps without periodic points are not too complex. Now we show that they are not chaotic in the sense of Li-Yorke and also in the sense of entropy.

Let (X, f) be a topological dynamical system. A pair $(x, y) \in X \times X$ is said to be *proximal* if $\liminf_{n\to+\infty} d(f^n x, f^n y) = 0$ and a pair such that $\lim_{n\to+\infty} d(f^n x, f^n y) = 0$ is said to be *asymptotic*. The sets of proximal pairs and asymptotic pairs of (X, f) are denoted by $Prox(X, f)$ and $Asym(X, f)$ respectively. A pair is said to be a *Li-Yorke pair* if it is proximal but not asymptotic. A set $S \subset X$ is called a *scrambled set* if any pair of distinct points in S is a Li-Yorke pair. (X, f) is *chaotic in the sense of Li-Yorke* if it admits a uncountable scrambled set.

Let (G, f) be a graph map without periodic points. Then from the proof of Theorem 4.1 and Theorem 4.2 one has $Prox(X, f) = Asym(X, f)$. This means f has no Li-Yorke pair. Hence one has the following proposition.

Proposition 5.4. *Let G be a graph and $f : G \to G$ a continuous map. If (G, f) has Li-Yorke pair, then $P(f) \neq \emptyset$. Especially, if (G, f) is chaotic in the sense of Li-Yorke, then $P(f) \neq \emptyset$.*

Proposition 5.5. *Let G be a graph and $f : G \to G$ a continuous map without periodic points. Then the topological entropy $h(f) = 0$.*

Proof. Since $h(f) = h(f|_{\Omega(f)})$, $h(f)$ is the same as the entropy of the irrational rotation of the unit circle or a Denjoy system by Corollary 4.4. But either of them has entropy zero, and one has $h(f) = 0$. □

Now we show the graph system (G, f) without periodic points is null, i.e. its sequence entropy is zero for any sequence. Especially, one gets Proposition 5.5 as its corollary. Firstly we give the definition of the sequence entropy. Let (X, f) be a dynamical system. Let $A = \{0 \leq t_1 < t_2 < \ldots\} \subseteq \mathbb{Z}_+$ and $\mathcal{U}$ be a finite open cover of X. The *topological sequence entropy of $\mathcal{U}$ with respect to (X, f) along A* is defined by $h_A(f, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n} f^{-t_i}\mathcal{U})$, where $N(\bigvee_{i=1}^{n} f^{-t_i}\mathcal{U})$ is the minimal cardinality among all cardinalities of subcovers of $\bigvee_{i=1}^{n} f^{-t_i}\mathcal{U}$. The *topological sequence entropy of* (X, f) along sequence A is $h_A(f) = \sup_{\mathcal{U}} h_A(f, \mathcal{U})$, where supremum is taken over all finite open covers of X. If $A = \mathbb{Z}_+$ one recovers standard topological entropy. In this case one omits the superscript $\mathbb{Z}_+$.

We give another definition for the sequence entropy. We say that a set $W \subseteq X$ (f, A, ε, n)*-spans* a set $B \subseteq X$ if for any $x \in B$ there is $y \in W$ such that $d(f^{t_i}x, f^{t_i}y) < \varepsilon$ for all $1 \leq i \leq n$, where $\varepsilon > 0$ and $n \in \mathbb{N}$. A subset of X is said to be a (f, A, ε, n)- *span* if it (f, A, ε, n)- spans X. Let $Span(f, A, \varepsilon, n)$ denote the smallest cardinalities of all (f, A, ε, n)-spans. Then one can prove that

$$h_A(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(f, A, \varepsilon, n).$$

For the details see [13].

A system (X, f) is *null* if its sequence entropy is zero for any sequence. In [15], authors showed that any minimal null system is an almost one to one extension of some equicontinuous system. In [14], author studied the topological sequence entropy for maps of the circle. He showed a circle map f has a Li-Yorke pair if and only if there is an sequence $A \subseteq \mathbb{Z}_+$ such that $h_A(f) > 0$. Here we will show that a graph map without periodic points is null.

Lemma 5.6. [11] *Let G be a graph and $f : G \to G$ a continuous map. If $Y = \bigcap_{n\geq 0} f^n(G)$ and A is any sequence of $\mathbb{Z}_+$, then $h_A(f) = h_A(f|_Y)$.*

Theorem 5.7. *Let G be a graph and $f : G \to G$ a continuous map without periodic points. Then (G, f) is null.*

Proof. By Lemma 5.6, we can assume that f is surjective. By Theorem 4.1, we only need to prove $(G_{\mathbf{\Gamma}}, \varphi)$ is null, where $(G_{\mathbf{\Gamma}}, \varphi)$ is the order 1 extension of the irrational rotation $h_r : C \to C$ (see Definition 3.3).

Let S be the unique minimal set of f. Then $h_A(f|_S) = 0$(see [17] or [14]). So

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(f|_S, A, \varepsilon, n) = 0.$$

For any fixed $\varepsilon > 0$, we are going to estimate $Span(f, A, \varepsilon, n)$. Let $G_1, G_2, \ldots, G_k$ be the all graphs in $\{G_t\}_{t\in T\cup T_0}$ with diameter greater than $\varepsilon/2$. Let W be a $(f|_S, A, \varepsilon/2, n)$- span. Take any point x whose $(f, A, n)-$ orbit $\{f^{t_i}x : 1 \leq i \leq n\}$ lies in $G_{\mathbf{\Gamma}} \setminus \bigcup_{i=1}^{k} G_i$. If $x \in S$, then x is (f, A, ε, n)- spanned by W. If $x \notin S$, then $x \in G_t$ for some $t \in T \cup T_0$. Let $y \in S \cap G_t$. Then $d_{\mathbf{\Gamma}}(f^{t_i}x, f^{t_i}y) \leq \varepsilon/2, \forall 1 \leq i \leq n$. Since $y \in S$ is $(f, A, \varepsilon/2, n)$-spanned by a point $z \in W$, x is (f, A, ε, n)-spanned by z.

Now it remains to consider the points whose (f, A, ε, n)- orbit meets $\bigcup_{i=1}^{k} G_i$. For any $1 \leq i \leq n$, we cut G_i into N_i segments and each segments shorter than $\varepsilon/2$. Let $G_{ij} = \{x \in G_G : f^{t_i}x \in G_j\}$, where $1 \leq i \leq n, 1 \leq j \leq k$. By the construction of $G_{\mathbf{\Gamma}}$, $G_{ij} \in \{G_t\}_{t\in T\cup T_0}$ and each element in its (f, A, n)- orbit $\{f^{t_1}G_{ij}, f^{t_2}G_{ij}, \ldots, f^{t_n}G_{ij}\}$ is either in $\{G_t\}_{t\in T\cup T_0}$ or a point in S. Hence at most k of them have diameter greater than $\varepsilon/2$. Note we have cut every G_i $(1 \leq i \leq k)$ into N_i segments with length less than $\varepsilon/2$ and we view the graph G_t with diameter less than $\varepsilon/2$ itself as one segment. Thus to each point $x \in G_{ij}$ we can assign its code the sequence $(S_1(x), S_2(x), \ldots, S_n(x))$, S_l is the segment containing $f^{t_l}x$. We have at most $N_1N_2 \ldots N_k$ different codes and all points with the same code can be $(f, A, \varepsilon/2, n)$- spanned by one point.

To sum up, we have $Span(f, A, \varepsilon, n) \leq Span(f|_S, A, \varepsilon, n) + n \cdot k \cdot N_1N_2 \ldots N_k$. So

$$h_A(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Span(f, A, \varepsilon, n) = 0.$$

The proof is completed. □

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