GRAPH MAPS WHOSE PERIODIC POINTS FORM A CLOSED SET

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Abstract. Let G be a graph, and $f : G \to G$ be a continuous map. Denote by $P(f)$, $R(f)$ and $\omega(f)$ the sets of periodic points, recurrent points and ω-limit points of f respectively. In this paper we show that if $P(f)$ is closed, then $\omega(f) = R(f)$.

1. Introduction

A *topological dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and f is a continuous map from X to itself. One uses $\mathbb{N}$ to denote the set of the natural numbers and $\mathbb{Z}_+$ the non-negative integers. For $x \in X$, $\{f^n(x) : n \in \mathbb{Z}_+\}$ is called the *orbit* of x and is denoted by $\mathcal{O}(x, f)$. x is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$. x is called a *recurrent point* of f if for any neighborhood U of x and any $m \in \mathbb{N}$ there exists $n > m$ such that $f^n(x) \in U$. Let $\omega(x)$ denote the set of ω-limit points of the orbit of x (precisely, $z \in \omega(x)$ if and only if some subsequence of the sequence $\{f^n(x)\}$ converges to z). Hence x is recurrent if and only if $x \in \omega(x)$. Set $\omega(f) = \bigcup_{x \in X} \omega(x)$. x is *non-wandering* if for any neighborhood U of x there is some $n \in \mathbb{N}$ such that $f^{-n}(U) \cap U \neq \emptyset$. Let $P(f)$, $R(f)$ and $\Omega(f)$ denote the sets of periodic points, recurrent points and non-wandering points of f respectively.

For $x, y \in X$ and $\varepsilon > 0$, an ε*-chain* of f from x to y is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ in X with $n > 0$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for $0 \le i < n$. We say that x is *chain recurrent* (*under* f) if for every $\varepsilon > 0$, there is an ε-chain from x to x. Denote the set of all chain recurrent points of f by $CR(f)$. For more about chain recurrence etc. please refer to [3].

By the definitions one can easily check the following inclusion relation

$$P(f) \subset R(f) \subset \omega(f) \subset \Omega(f) \subset CR(f). \tag{1.1}$$

It is easy to give an interval map $f : I \to I$ such that

$$P(f) \subsetneq R(f) \subsetneq \omega(f) \subsetneq \Omega(f) \subsetneq CR(f).$$

Thus, in general, no inclusion symbol "$\subset$" in (1.1) can be replaced by the equality "$=$".

Block [2] showed that if $f : I \to I$ is an interval map and $P(f)$ is a finite set consisting only of fixed points, then $\Omega(f) = P(f)$. Coven and Hedlund [9] extended this, obtaining the same conclusion from the weaker hypothesis that some power $g = f^n$ of f simultaneously fixes all the periodic points, and they also proved that if $P(f)$ is closed, then $P(f) = R(f)$. Nitecki [13] and Xiong [16] proved independently that if

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the set of periodic points is closed, then every non-wandering point is periodic, i.e. $P(f) = \Omega(f)$. Block and Franks [5] improved this result, and showed that under the same hypotheses every chain recurrent point is periodic. That is, if $P(f)$ is closed in I, then $P(f) = R(f) = \omega(f) = \Omega(f) = CR(f)$.

For a circle map f, Block et al [4] proved that $P(f) = \Omega(f)$ if and only if $P(f)$ is closed and nonempty, and Block and Franke [6] also obtained some necessary and sufficient conditions for $P(f) = CR(f)$. In [7, 8] Blokh constructed the "spectral" decomposition of the sets $\overline{P(f)}, \omega(f)$ and $\Omega(f)$ for a graph map f, and obtained a series of applications of the "spectral" decomposition.

In this note we will study graph maps with the sets of periodic points being closed. Our main result is the following theorem:

Theorem 2.9. *Let $f : G \to G$ be a graph map. If the set $P(f)$ of periodic points of f is closed in G, then $\omega(f) = R(f)$.*

In addition, we will give an example to show that, for graph maps f with $P(f)$ being closed, if we do not put any additional condition, then the conclusion $\omega(f) = R(f)$ in Theorem 2.9 can not be strengthened to $\Omega(f) = R(f)$ or $\omega(f) = P(f)$.

2. Graph maps whose periodic points form a closed set

First recall some notions about graphs. A metric space X is called an *arc* (resp. an *open arc*, an *circle*) if it is homeomorphic to the interval $[0, 1]$ (resp. the open interval $(0, 1)$, the unit circle S^1). Let A be an arc and $h : [0, 1] \to A$ be a homeomorphism. The set of endpoints of A is $\partial A = \{h(0), h(1)\}$. A metric space G is called a *graph* if there are finitely many arcs $A_1, \ldots, A_n$ ($n \geq 1$) in G such that $G = \bigcup_{i=1}^n A_i$ and $A_i \cap A_j = \partial A_i \cap \partial A_j$ for all $1 \leq i < j \leq n$.

Let G be a graph. The set $V(G)$ of *vertexes* of G is a given finite subset of G such that: (1) every connected component E of $G - V(G)$ is an open arc, and the closure $\overline{E}$ of E is an arc; (2) for any circle C in G, $C \cap V(G)$ contains at least three points. Every connected component E of $G - V(G)$ is called an *edge* of G. A continuous map from a graph to itself is called a *graph map*.

The following lemma is well known:

Lemma 2.1. *Let (X, f) be a topological dynamical system. Then $f(\overline{R(f)}) \subset \overline{R(f)}$, and $f^{-1}(x) \cap \omega(f) \neq \emptyset$ for any $x \in \omega(f)$.*

Let E be an edge of a graph G. An ordering $\prec$ on E is called a *natural ordering* if there is a homeomorphism $h : (0, 1) \to E$ such that $h(r) \prec h(s)$ if and only if $0 < r < s < 1$. Denote by $\succ$ the inverse ordering of $\prec$. From Lemma 2.1 we obtain immediately

Lemma 2.2. *Let $f : G \to G$ be a graph map. If $\omega(f) - \overline{R(f)} \neq \emptyset$, then there exist an edge E of G with a natural ordering $\prec$ and points $\{w_0, w_1, \ldots\} \subset E \cap \big(\omega(f) - \overline{R(f)}\big)$ such that $w_n \in \mathcal{O}(w_{n+1}, f)$ for all $n \geq 0$, and*

$$w_0 \prec w_1 \prec w_2 \prec \ldots \quad \text{or} \quad w_0 \succ w_1 \succ w_2 \succ \ldots .$$

Let X be a metric space, $f : X \to X$ be a continuous map and A, A' be two arcs in X. If there exist a subarc A_0 of A and $n \in \mathbb{N}$ such that $f^n(A_0) = A'$, then we write $A \stackrel{f}{\Longrightarrow} A'$. The following lemma is well known.

Lemma 2.3. *Let X be a metric space, $f : X \to X$ be a continuous map and A, A' be two arcs in X. If $A \overset{f}{\Longrightarrow} A' \overset{f}{\Longrightarrow} A$, then $P(f) \cap A \neq \emptyset$.*

The following lemma is the key result of the paper. With the help of this lemma, we can use the main result of [5] to get Theorem 2.9.

Lemma 2.4. *Let $f : G \to G$ be a graph map. If $\omega(f) - \overline{R(f)} \neq \emptyset$, then there exist an arc $A \subset G$ and $m \in \mathbb{N}$ such that*

$$f^m(A) \subset A \quad \text{and} \quad A \cap \big(\omega(f) - \overline{R(f)}\big) \neq \emptyset.$$

Proof. By Lemma 2.2, there is an edge E of G with a natural ordering $\prec$ and $\{w', w, w''\} \subset E \cap \Big(\omega(f) - \overline{R(f)}\Big)$ such that $w \in \mathcal{O}(w', f)$, $w'' \in \mathcal{O}(w, f)$ and

$$w' \prec w \prec w'' \quad \text{or} \quad w'' \prec w \prec w'.$$

For the convenience of statement, we may assume that $E = (0, 1)$ with endpoints $\{0, 1\} \subset V(G)$. Suppose that $w = f^{m_1}(w')$ and $w'' = f^{m_2}(w)$ for some $m_1, m_2 \in \mathbb{N}$. Take open intervals J', J and J'' in $E = (0, 1)$ such that $w' \in J'$, $w \in J$, $w'' \in J''$,

$$f^{m_1}(J') \subset J, \quad f^{m_2}(J) \subset J'', \quad J' \cap J = J \cap J'' = \emptyset,$$
$$(J' \cup J \cup J'') \cap \overline{R(f)} = \emptyset, \quad (J' \cup J \cup J'') \cap \mathcal{O}(w', f) = \{w', w, w''\}.$$

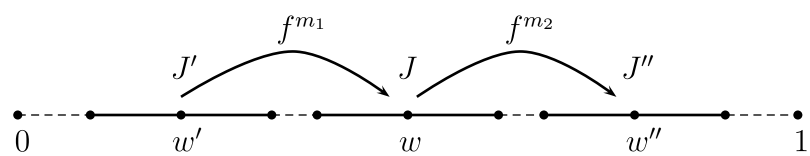

Figure 1. The case $w' \prec w \prec w''$

Take $\{x'_i : i \in \mathbb{N}\} \subset J'$, $\{x_i : i \in \mathbb{N}\} \subset J$ and $\{x''_i : i \in \mathbb{N}\} \subset J''$ satisfying

(C.1) $(x'_1, x'_2, x'_3, \ldots), (x_1, x_2, x_3, \ldots)$ and $(x''_1, x''_2, x''_3, \ldots)$ are strictly monotonic sequences in $E = (0, 1)$, and $|x'_i - w'| \to 0$ as $i \to \infty$;

(C.2) $f^{m_1}(x'_i) = x_i$ and $f^{m_2}(x_i) = x''_i$ for all $i \in \mathbb{N}$;

(C.3) For each $i \in \mathbb{N}$ there is a $k_i > m_1 + m_2$ such that $f^{k_i}(x'_i) = x'_{i+1}$.

For convenience, we may assume that the middle sequence $(x_1, x_2, \ldots)$ is a strictly increasing sequence, and then we write

$$\text{(2.1)} \qquad \begin{cases} J_1 = J'',\ w_1 = w'',\ \text{and } y_i = x''_i \text{ for all } i \in \mathbb{N}, & \text{if } w' < w < w''; \\ J_1 = J',\ w_1 = w',\ \text{and } y_i = x'_i \text{ for all } i \in \mathbb{N}, & \text{if } w'' < w < w'. \end{cases}$$

For each $i \in \mathbb{N}$, let

$$\text{(2.2)} \qquad n_i = \begin{cases} k_i - m_2, & \text{if } w' < w < w''; \\ k_i + m_1, & \text{if } w'' < w < w'. \end{cases}$$

It follows from (C.2), (C.3) and (2.2) that

$$\text{(2.3)} \qquad f^{n_i}(y_i) = x_{i+1}, \text{ for all } i \in \mathbb{N}.$$

Let $g = f^{k_1}$. Then $g(x_1) = x_2 > x_1$. Let

$$K_0 = [x_1, x_2],\ \ K_i = \bigcup_{\lambda=0}^{i} g^{\lambda}(K_0) \text{ for all } i \in \mathbb{N}, \text{ and } K_\infty = \bigcup_{i=0}^{\infty} K_i.$$

Then K_i and K_∞ are connected subsets of G, K_i is closed and $g(K_\infty) \subset K_\infty$, $g(\overline{K}_\infty) \subset \overline{K}_\infty$.

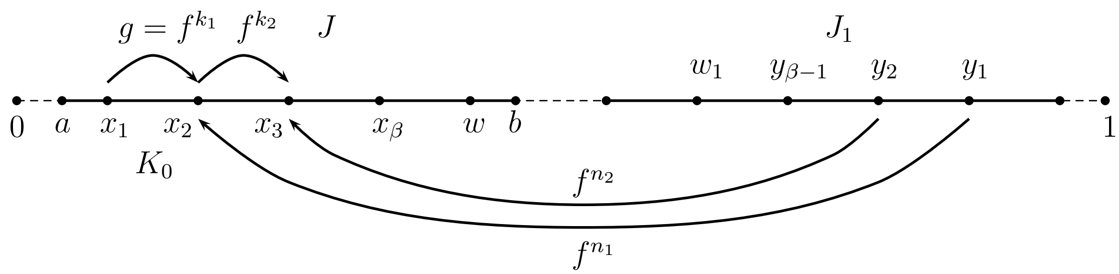

Figure 2.

Claim 1. *If there are $c \in (x_2, 1]$ and $n \geq 1$ such that $c \notin K_n$, then $K_n \subset [x_1, c)$, and for each $i \in \{1, 2, \ldots, n\}$,*

(2.4) $$t < g^i(t) < c \quad \text{for all } \ t \in [x_1, x_2].$$

Proof of Claim 1. Note that

$$[x_1, x_2] \bigcap \left(\bigcup_{i=1}^{\infty} \mathrm{Fix}(g^i) \right) = [x_1, x_2] \cap P(g) \subset \overline{J} \cap \overline{R(f)} = \emptyset.$$

It follows from $g(x_1) > x_1$ that (2.4) holds for $i = 1$. If $n \geq 2$ and (2.4) holds for some $i = i_0 \in \{1, 2, \ldots, n-1\}$, then from $x_1 < x_2 < g^{i_0}(x_2) = g^{i_0+1}(x_1) < c$ we see that (2.4) also holds for $i = i_0 + 1$. Thus (2.4) holds for all $i \in \{1, 2, \ldots, n\}$, and hence $K_n \subset [x_1, c)$. This completes the proof of Claim 1.

Suppose that the endpoints of J are a and b with $a < b$, that is, $J = (a, b)$.

Claim 2. $[x_1, b] \subset K_\infty$.

Proof of Claim 2. If $[x_1, b] \not\subset K_\infty$, then there is $c \in (x_2, , b]$ such that $c \notin K_\infty$. By Claim 1. $\overline{K}_\infty$ is a closed interval contained in $[x_1, c]$. It follows from $g(\overline{K}_\infty) \subset \overline{K}_\infty$ that

$$R(f) \cap \overline{J} \supset \mathrm{Fix}(g) \cap \overline{J} \supset \mathrm{Fix}(g) \cap \overline{K}_\infty \neq \emptyset.$$

But this will lead to a contradiction. Thus Claim 2. holds.

Claim 3. *There exist $\alpha \in \mathbb{N}$ and a closed interval $[v_1, v_2] \subset [x_1, x_2)$ such that $g^\alpha(v_1) = g^\alpha(x_1) \in [x_2, w)$, $g^\alpha(v_2) = w$, and $g^\alpha([v_1, v_2]) = [g^\alpha(x_1), w]$.*

Proof of Claim 3. By Claim 2. there exists $\alpha \in \mathbb{N}$ such that $w \in K_\alpha$ and $w \notin K_{\alpha-1}$. Hence, by (2.4), $x_2 \leq g^\alpha(x_1) = g^{\alpha-1}(x_2) < w$. Let $v_2 = \min\{t \in [x_1, x_2] : g^\alpha(t) = w\}$, and let $v_1 = \max\{t \in [x_1, v_2) : g^\alpha(t) = g^\alpha(x_1)\}$. Then $[v_1, v_2]$ satisfies the conditions in Claim 3.

Claim 4. *Let* α *and* $[v_1, v_2]$ *be as in Claim 3. Take an integer* $\beta \geq 2$ *such that* $x_\beta \in [g^\alpha(x_1), w)$. *Let* $\psi = f^{k_\beta}$, $L_0 = [x_\beta, w]$, $L_i = \bigcup_{\lambda=0}^{i} \psi^\lambda(L_0)$ *for each* $i \in \mathbb{N}$, $L_\infty = \bigcup_{i=0}^{\infty} L_i$, *and let* y_i *be defined as in (2.1). Then* L_∞ *is a connected set contained in* $[x_\beta, y_{\beta-1})$.

Proof of Claim 4. It follows from $\psi(x_\beta) = x_{\beta+1} \in (x_\beta, w)$ that $L_0 \cap \psi(L_0) \neq \emptyset$. Thus each L_i and L_∞ are connected subsets of G.

If $y_{\beta-1} \in L_\infty$, then there exist $j \in \mathbb{N}$ and $z \in L_0$ such that $\psi^j(z) = y_{\beta-1}$. Write $\mu = jk_\beta + n_{\beta-1}$. By (2.3), we have $f^\mu(z) = f^{n_{\beta-1}}\psi^j(z) = f^{n_{\beta-1}}(y_{\beta-1}) = x_\beta$. Since $f^\mu(w) \notin J$, and $f^\mu(L_0)$ is a connected set containing x_β and $f^\mu(w)$, there exists a closed subinterval L_0' of L_0 such that

$$f^\mu(L_0') = K_0 \quad \text{or} \quad f^\mu(L_0') = L_0.$$

This means that

$$L_0 \overset{f}{\Longrightarrow} K_0 \quad \text{or} \quad L_0 \overset{f}{\Longrightarrow} L_0. \tag{2.5}$$

By Claim 3. we have $K_0 \overset{f}{\Longrightarrow} L_0$. Hence, by Lemma 2.1, from (2.5) we get $P(f) \cap L_0 \neq \emptyset$. But this contradicts that $P(f) \cap J = \emptyset$. Thus it must hold that $y_{\beta-1} \notin L_\infty$. Similar to Claim 1. from $y_{\beta-1} \notin L_\infty$ we can derive $L_\infty \subset [x_\beta, y_{\beta-1})$. The proof of Claim 4. is completed.

we now put $A = \overline{L}_\infty$. Then $A \cap \left(\omega(f) - \overline{R(f)}\right) \supset \{w\} \neq \emptyset$. By Claim 4. A is a closed interval contained in $[x_\beta, y_{\beta-1}]$. Let $m = k_\beta$. Then $f^m(A) = \psi(A) \subset A$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let* $\varphi : [0, 1] \to [0, 1]$ *be an interval map. If the set* $P(\varphi)$ *of periodic points of* φ *is closed, then* $\omega(\varphi) = \overline{R(\varphi)}$.

Proof. If $P(\varphi)$ is closed, then by the main result of [5] one has $CR(\varphi) = P(\varphi)$. Noting that $P(\varphi) \subset \omega(\varphi) \subset CR(\varphi)$ and $P(\varphi) \subset \overline{R(\varphi)} \subset CR(\varphi)$, we obtain that $\omega(\varphi) = \overline{R(\varphi)}$. □

Proposition 2.6. *Let* $f : G \to G$ *be a graph map. If* $\omega(f) - \overline{R(f)} \neq \emptyset$, *then* $P(f)$ *is not closed.*

Proof. Let the arc $A \subset G$ and $m \in \mathbb{N}$ be as in Lemma 2.4. Let $\varphi = f^m|_A : A \to A$. Then φ can be regarded as an interval map, and we have $P(\varphi) = P(f^m) \cap A = P(f) \cap A$, $R(\varphi) = R(f^m) \cap A = R(f) \cap A$, and $\omega(\varphi) \subset \omega(f^m) \cap A = \omega(f) \cap A$. Since $A \cap \overline{G - A}$ is a finite set, it is easy to check that $\overline{R(\varphi)} = \overline{R(f)} \cap A$ and $\omega(\varphi) = \omega(f) \cap A$. Thus, by Lemma 2.4, $\omega(\varphi) - \overline{R(\varphi)} \neq \emptyset$, and hence by Lemma 2.5, $P(\varphi)$ is not closed. This with $P(\varphi) = P(f) \cap A$ implies that $P(f)$ is not closed. □

The following two lemmas are known.

Lemma 2.7. ([12, Corollary 2]). *Let* $f : G \to G$ *be a graph map. Then* $\omega(f)$ *is closed in* G, *and hence* $\overline{R(f)} \subset \omega(f)$.

Lemma 2.8. ([11, Corollary 2.4]). *Let* $f : G \to G$ *be a graph map. If* $P(f)$ *is closed in* G, *then* $R(f)$ *is closed.*

In fact, Sharkovskii [15] has shown that Lemma 2.7 is true for interval maps. In the proof of [7, Theorem 4], Blokh pointed out that by the same methods as [15] one can easily prove that Lemma 2.7 is true for graph maps. In addition, by means of the main result of [10] one can also give a simple proof of Lemma 2.7.

From Proposition 2.6, Lemma 2.7 and Lemma 2.8, we obtain the following theorem readily, which is the main result of this paper.

Theorem 2.9. *Let $f : G \to G$ be a graph map. If the set $P(f)$ of periodic points of f is closed in G, then $\omega(f) = R(f)$.*

The following example shows that Theorem 2.9 cannot be strengthened to be $\Omega(f) = R(f)$ or $\omega(f) = P(f)$.

Example 2.10. Now we construct a graph map of which the set of periodic points is closed. Let $S^1 = \{e^{2\pi \mathbf{i} t} \in \mathbb{C} : t \in \mathbb{R}\}$ be the unit circle in the complex plane $\mathbb{C}$, and let $G_1 = S^1 \cup [-1, 1]$ be a graph in $\mathbb{C}$. Note that the interval $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}$. Define $f_1 : G_1 \to G_1$ by

$$\begin{cases} f_1(e^{2\pi \mathbf{i} t}) = e^{4\pi \mathbf{i} t}, & \text{if } t \in [0, 1/2]; \\ f_1(e^{2\pi \mathbf{i} t}) = 1, & \text{if } t \in [1/2, 1]; \\ f_1(r) = e^{\pi \mathbf{i}(1-|r|)}, & \text{if } r \in [-1, 1]. \end{cases}$$

It is easy to see that f_1 is continuous, and

$$P(f_1) = R(f_1) = \omega(f_1) = \{1\} \subsetneq \Omega(f_1) = \{-1, 1\}.$$

Let $G_2 = \{4 + z : z \in S^1\}$ be the circle in $\mathbb{C}$ with center 4 and radius 1, and let $f_2 : G_2 \to G_2$ be an irrational rotation. Then (for example see [1, 3, 14])

$$P(f_2) = \emptyset \subsetneq R(f_2) = \omega(f_2) = \Omega(f_2) = G_2.$$

Let $G = G_1 \cup [1, 3] \cup G_2$ (see Figure 3.). Define a continuous map $f : G \to G$ such that $f(z) = f_i(z)$ if $z \in G_i$ with $i \in \{1, 2\}$; $f(r) = 2r - 1$ if $r \in [1, 2]$; and $f([2, 3]) \subset G_2$. Then

$$P(f) = \overline{P(f)} = P(f_1) = \{1\} \subsetneq R(f) = \omega(f) = \{1\} \cup G_2 \subsetneq \Omega(f) = \{-1, 1\} \cup G_2.$$

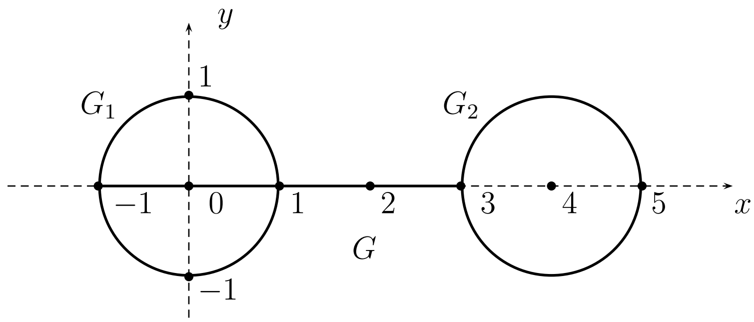

Figure 3.

From Example 2.10 we see that in general for graph maps the conclusion in Theorem 2.9 cannot be strengthened to be $\Omega(f) = R(f)$ or $\omega(f) = P(f)$.

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