PROXIMITY AND DISTALITY VIA FURSTENBERG FAMILIES

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Abstract. In this paper proximity, distality and recurrence are studied via Furstenberg families. A new proof of some classical results on the conditions when a proximal relation is an equivalence one is given. Moreover, for a family $\mathcal{F}$, $\mathcal{F}$-almost distality and $\mathcal{F}$-semi-distality are defined and characterized. As an application a new characterization of PI-flows is obtained.

1. Introduction

Throughout this paper a *topological dynamical system* (TDS for short) is a pair (X, T), where X is a nonvoid compact metric space with a metric d and T is a continuous surjective map from X to itself. We use $\mathbb{Z}$ to denote the set of integers, $\mathbb{Z}_+$ the set of non-negative integers and $\mathbb{N}$ the set of natural numbers. Let $Trans_T = \{x : \omega T(x) = X\}$, where $\omega T(x)$ is the ω-limit set of x. Say (X, T) is *transitive* if $Trans_T \neq \emptyset$. In fact, $Trans_T$ is a dense G_δ set when it is not empty. Say (X, T) is *minimal* if X is the only non-empty closed and invariant subset, and $x \in X$ is a *minimal point* if it belongs to some minimal subsystem of X. Sometimes we need to consider the case when the phase space is an arbitrary compact Hausdorff space, and in this case we define minimality in the same way.

Classically, one way of studying a TDS is to consider the asymptotic behavior of pairs of points. A pair $(x, y) \in X \times X = X^2$ is said to be *proximal* if $\liminf_{n\to+\infty} d(T^n x, T^n y) = 0$ and the one with $\lim_{n\to+\infty} d(T^n x, T^n y) = 0$ is said to be *asymptotic*. If in addition $x \neq y$, then the pair (x, y) is said to be *proper*. The sets of proximal pairs and asymptotic pairs of (X, T) are denoted by $P(X, T)$ and $Asym(X, T)$ respectively. $P(X, T)$ is a reflexive, symmetric, T-invariant relation, but in general not transitive or closed [3, 4, 5].

A pair $(x, y) \in X^2$ which is not proximal is said to be *distal*. A pair is a *Li-Yorke pair* if it is proximal but not asymptotic. $x \in X$ is a *recurrent point* if there is an increasing sequence $\{n_i\}$ of $\mathbb{N}$ with $T^{n_i} x \to x$. A pair $(x, y) \in X^2 \setminus \Delta_X$ is a *strong Li-Yorke pair* if it is proximal and is a recurrent point of $T \times T$. It is easy to check that a strong Li-Yorke pair is a Li-Yorke pair. A system without proper proximal pairs (Li-Yorke pairs, strong Li-Yorke pairs) is called *distal* (*almost distal*, *semi-distal* respectively). It is clear that a distal system is almost distal and an almost distal system is semi-distal.

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A beautiful characterization of distality was given by R. Ellis using so-called enveloping semigroup. Given a TDS (X,T) its *enveloping semigroup* $E(X,T)$ is defined as the closure of the set $\{T^n : n \in \mathbb{Z}_+\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). Ellis showed that a TDS (X,T) is distal iff $E(X,T)$ is a group iff every point in X^2 is minimal [9]. The notion of almost distal was first introduced by Blandchard etc. [7]. Let the *adherence semigroup* $\mathcal{H}(X,T)$ be $limsup\{T^n\} = \bigcap_{k=1}^{\infty} \overline{\{T^n : n = k, k+1, \cdots\}} \subset X^X$. They showed that a TDS (X,T) is almost distal iff $(\mathcal{H}(X,T),T)$ is minimal iff every ω-limit set in $(X^2, T\times T)$ is minimal. Recently, Akin etc. studied distality concepts for Ellis actions [1]. They defined a system without strong Li-Yorke pairs to be *semi-distal*, i.e. every $(x,y) \in X^2$ which is both proximal and recurrent is in the diagonal. They gave an elegant characterization of semi-distality via the enveloping semigroup, namely they showed that a TDS is semi-distal iff every idempotent in $\mathcal{H}(X,T)$ is minimal iff every recurrent point in $(X^2, T\times T)$ is minimal.

In this paper we investigate the proximal relation from the viewpoint of Furstenberg families and give a new proof of some classical results on the conditions when a proximal relation is an equivalence one. By using the family notion our proofs become simpler and clearer. Moreover, family machinery is applied to describe family versions of distality, almost distality and semi-distality. Different notions are unified by this family viewpoint, and in particular, we show that a minimal PI-flow can be viewed as some kind of semi-distal one. By applying the structure theorems of some special minimal systems, we can give a negative answer to a conjecture by Blanchard etc. [7] on the structure of minimal almost distal systems.

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2. Preliminary

Firstly we introduce some notations related to a family (for details see [2, 10]). Let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upwards, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that a family $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ can generate a family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$. If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a *filter*. For a family $\mathcal{F}$, the *dual family* is

$$k\mathcal{F} = \{F \in \mathcal{P} | F \cap F' \neq \emptyset \ for\ all\ F' \in \mathcal{F}\}. \tag{1}$$

It is easy to see that $k\mathcal{F}$ is a family, proper if $\mathcal{F}$ is. Clearly, $k(k\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $k\mathcal{F}_2 \subset k\mathcal{F}_1$. For families $\mathcal{F}_1$ and $\mathcal{F}_2$, let $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 | F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. Thus we have $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$. It is easy to check that $\mathcal{F}$ is a filter iff $\mathcal{F} = \mathcal{F}\cdot\mathcal{F}$.

For $i \in \mathbb{Z}$ and $F \subseteq \mathbb{Z}_+$ let $F + i = \{j + i : j \in F\} \cap \mathbb{Z}_+$. A family $\mathcal{F}$ is called *translation* $\pm$ *invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ we have $F \pm i \in \mathcal{F}$. A family $\mathcal{F}$

is called *translation invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ iff $F - i \in \mathcal{F}$. For a family $\mathcal{F}$ let

$$\tau\mathcal{F} = \{F \in \mathcal{P} : \bigcap_{j=1}^{n}(F - i_j) \in \mathcal{F} \text{ for } n \in \mathbb{N} \text{ and each } \{i_1, i_2, \cdots, i_n\} \subset \mathbb{Z}_+\}. \tag{2}$$

$\mathcal{F}$ is a *thick family* if $\tau\mathcal{F} = \mathcal{F}$, and it is easy to see $\tau\mathcal{F}$ is the largest thick family contained in $\mathcal{F}$. By the definition a filter $\mathcal{F}$ is $-$translation invariant iff it is thick.

Now let us recall some important sets and families. Let $\mathcal{B}$ the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that $\mathcal{B}$ is the largest proper translation invariant family and its dual $k\mathcal{B}$, the family of cofinite subset, is the smallest one. A subset F of $\mathbb{Z}_+$ is *thick* if $F \in \tau\mathcal{B}$, equivalently, F is thick if and only if it contains arbitrarily long runs of positive integers. Each element of $k\tau\mathcal{B}$ is said to be *syndetic* or *relatively dense*. F is syndetic if and only if there is N such that $\{i, i+1, \cdots, i+N\} \cap F \neq \emptyset$ for every $i \in \mathbb{Z}_+$. A set in $\tau k\tau\mathcal{B}$ is called *replete* or *thickly syndetic*. $F \in \tau k\tau\mathcal{B}$ if and only if for every N the positions where length N runs begin form a syndetic set. The set in $k\tau k\tau\mathcal{B}$ is called *big* or *piecewise syndetic*. $F \in k\tau k\tau\mathcal{B}$ if and only if it is the intersection of a thick set and a syndetic set. All of these families are translation invariant, and $\tau k\tau\mathcal{B}$ is a filter.

A family $\mathcal{F}$ is *full* if $\mathcal{F} \cdot k\mathcal{F} \subset \mathcal{B}$. If $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F} \subset \mathcal{B}$. If $\mathcal{F}$ is a filter, then $k\mathcal{B} \subset \mathcal{F}$ implies $\mathcal{F}$ is full.

Let (X, T) be a dynamical system and $A, B \subset X$. We define the *hitting time set*

$$N_T(A, B) = N(A, B) = \{n \in \mathbb{Z}_+ : T^n(A) \cap B \neq \emptyset\}. \tag{3}$$

Especially, $N(x, B) = \{n \in \mathbb{Z}_+ : T^n x \in B\}$.

Now we generalize the notion of ω-limit set. Let (X, T) be a TDS and $\mathcal{F}$ be a family. Define

$$\omega_{\mathcal{F}}(T, x) = \omega_{\mathcal{F}}T(x) = \bigcap_{F \in k\mathcal{F}} \overline{T^F(x)} \tag{4}$$

where $T^F = \bigcup\{T^n | n \in F\}$.

By the definition $y \in \omega_{\mathcal{F}}T(x)$ iff $N(x, U) \in \mathcal{F}$ for every neighborhood U of y. When $\mathcal{F} = \mathcal{B}$, it is the usual ω-limit set, i.e. $\omega_{\mathcal{B}}T(x) = \omega T(x)$. It is easy to see that when $\mathcal{F}$ is translation + invariant $\omega_{\mathcal{F}}(T, x)$ is a closed invariant subset of X, i.e. $T\omega_{\mathcal{F}}(T, x) \subseteq \omega_{\mathcal{F}}(T, x)$.

Now we consider the *Stone$-\check{C}$ech* compactification of the semigroup $\mathbb{Z}_+$ with the discrete topology. The set of all ultrafilters on $\mathbb{Z}_+$ is denoted by $\beta\mathbb{Z}_+$. Let $A \subset \mathbb{Z}_+$ and define $\overline{A} = \{p \in \beta\mathbb{Z}_+ : A \in p\}$. It is easy to see that $\overline{A} \cap \overline{B} = \overline{A \cap B}, \overline{A} \cup \overline{B} = \overline{A \cup B}$, where $A, B \subset \mathbb{Z}_+$. The set $\{\overline{A} : A \subset \mathbb{Z}_+\}$ forms a basis for the open sets (and also a basis for closed sets) of $\beta\mathbb{Z}_+$. Under this topology, $\beta\mathbb{Z}_+$ is a compact Hausdorff space. Define $j : \mathbb{Z}_+ \to \beta\mathbb{Z}_+$ by $j(t) = \{A \subset \mathbb{Z}_+ : t \in A\}$. Then $(j, \beta\mathbb{Z}_+)$ is the maximum compactification of $\mathbb{Z}_+$, called *Stone $-$ $\check{C}$ech compactification* (for details see [4, 9]).

For $F \subset \mathbb{Z}_+$ the *hull of* F is $h(F) = \overline{F} = \{p \in \beta\mathbb{Z}_+ : F \in p\}$. For the family $\mathcal{F}$, the *hull of* $\mathcal{F}$ is defined by

$$h(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} h(F) = \bigcap_{F\in\mathcal{F}} \overline{F} = \{p \in \beta\mathbb{Z}_+ : \mathcal{F} \subseteq p\} \subseteq \beta\mathbb{Z}_+. \tag{5}$$

Given $A \subset \beta\mathbb{Z}_+$ define the *kernel of* A by

$$K(A) = \bigcap_{p\in A} p\ . \tag{6}$$

$K(A)$ is a filter on $\mathbb{Z}_+$. From the operators h and K, we obtain a one-to-one corresponding between the set of filters on $\mathbb{Z}_+$ and the set of closed subsets of $\beta\mathbb{Z}_+$ [9, 12, 2].

Let X be a compact metric space and S a semigroup. Let $\Phi : S \times X \to X$ be an action, i.e. for any $p, q \in S$, $\Phi^p \circ \Phi^q = \Phi^{pq}$. For $(p, x) \in S \times X$, denote

$$px = \Phi(p, x) = \Phi^p(x) = \Phi_x(p). \tag{7}$$

$\Phi^\# : S \to X^X$ is defined by $p \mapsto \Phi^p$. Hence $px = \Phi^\#(p)(x)$. An *Ellis semigroup* S is a compact Hausdorff semigroup such that the right translation map $R_p : S \longrightarrow S$, $q \longmapsto qp$ is continuous for every $p \in S$. An *Ellis action* of an Ellis semigroup S on a space X is a map $\Phi : S \times X \to X$ which is an action such that the adjoint map $\Phi^\#$ is continuous, or equivalently, Φ_x is continuous for each $x \in X$.

Now let (X, T) be a TDS. Then $\Phi : \mathbb{Z}_+ \times X \to X, (n, x) \mapsto T^n x$ is an action and it can be extended to an Ellis action $\Phi : \beta\mathbb{Z}_+ \times X \to X$. Hence we have a continuous map $\Phi^\# : \beta\mathbb{Z}_+ \to X^X$.

Define

$$H(\mathcal{F}) = H(X, \mathcal{F}) = \Phi^\#(h(\mathcal{F})) \subset X^X. \tag{8}$$

It is easy to see that for a family $\mathcal{F}$, $H(\mathcal{F}) \neq \emptyset$ iff $\mathcal{F}$ has finite intersection property.

Proposition 2.1. *Let (X, T) be a TDS and $\mathcal{F}$ be a filter. Then*

1. $H(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} \overline{T^F} \subseteq X^X$.
2. $\omega_{k\mathcal{F}} T(x) = H(\mathcal{F})x$.

Proof. (1). First we show that $\Phi^\#(\bigcap_{F\in\mathcal{F}} \overline{F}) = \bigcap_{F\in\mathcal{F}} \Phi^\#(\overline{F})$.

It is obvious that $\Phi^\#(\bigcap_{F\in\mathcal{F}} \overline{F}) \subseteq \bigcap_{F\in\mathcal{F}} \Phi^\#(\overline{F})$. Now let $p \in \bigcap_{F\in\mathcal{F}} \Phi^\#(\overline{F})$, i.e. $p \in \Phi^\#(\overline{F})$ for every $F \in \mathcal{F}$. Thus $(\Phi^\#)^{-1}(p) \cap \overline{F} \neq \emptyset$ for any $F \in \mathcal{F}$. Since $\mathcal{F}$ has finite intersection property, so does $\{(\Phi^\#)^{-1}(p) \cap \overline{F} : F \in \mathcal{F}\}$. As $\beta\mathbb{Z}_+$ is compact, we have $\bigcap_{F\in\mathcal{F}} ((\Phi^\#)^{-1}(p) \cap \overline{F}) \neq \emptyset$, i.e. $(\Phi^\#)^{-1}(p) \cap \bigcap_{F\in\mathcal{F}} \overline{F} \neq \emptyset$. That is $p \in \Phi^\#(\bigcap_{F\in\mathcal{F}} \overline{F})$.

Now we show $\Phi^\#(\overline{F}) = \overline{T^F}$. First by the continuity of $\Phi^\#$ we have $\Phi^\#(\overline{F}) \subseteq \overline{\Phi^\#(F)} = \overline{T^F}$. As $\Phi^\#(\overline{F})$ is compact, it is closed. And hence we have $\overline{T^F} = \overline{\Phi^\#(F)} \subseteq \overline{\Phi^\#(\overline{F})} = \Phi^\#(\overline{F})$. So, it follows that $\Phi^\#(\overline{F}) = \overline{T^F}$.

By the above two equations we have

$$H(\mathcal{F}) = \Phi^{\#}(h(\mathcal{F})) = \Phi^{\#}(\bigcap_{F\in\mathcal{F}} \overline{F}) = \bigcap_{F\in\mathcal{F}} \Phi^{\#}(\overline{F}) = \bigcap_{F\in\mathcal{F}} (\overline{T^F}) \subseteq X^X.$$

(2). Similar to (1), we have

$$H(\mathcal{F})x = \Phi^{\#}(h(\mathcal{F}))x = \Phi_x(h(\mathcal{F})) = \Phi_x(\bigcap_{F\in\mathcal{F}} \overline{F}) = \bigcap_{F\in\mathcal{F}} (\overline{T^F x}) = \omega_{k\mathcal{F}}T(x).$$

□

Remark 2.2. *Let (X,T) be a TDS. Then*

1. $E(X,T) = H([\mathbb{Z}_+]) = \Phi^{\#}(\beta\mathbb{Z}_+)$ *is the enveloping semigroup of* (X,T)*.*
2. $\mathcal{H}(X,T) = H(k\mathcal{B}) = \Phi^{\#}(\beta^* T)$*, where* $\beta^*\mathbb{Z}_+ = \beta\mathbb{Z}_+ \setminus \mathbb{Z}_+$*, is the adherence semigroup of* T*.*

Let $\pi : (X,T) \to (Y,S)$ be a factor map. Then there is a unique continuous semigroup homomorphism $\phi : E(X,T) \to E(Y,S)$ such that $\pi(px) = \phi(p)\pi(x)$, $x \in X, p \in E(X,T)$. We can get ϕ as follows. Let $\phi : \{T^n : n \in \mathbb{Z}_+\} \to \{S^n : n \in \mathbb{Z}_+\}$, $T^n \mapsto S^n$, where $\{T^n : n \in \mathbb{Z}_+\}, \{S^n : n \in \mathbb{Z}_+\}$ with the topology inherited from X^X and Y^Y. Then ϕ is uniformly continuous. And hence has a continuous extension, still called ϕ, to a continuous map of $E(X,T)$ to $E(Y,S)$. ϕ has the required properties. That is, we have:

Proposition 2.3. *Let (X,T) and (Y,S) be TDS. If $\pi : (X,T) \to (Y,S)$ be a factor map, then there is a unique continuous semigroup homomorphism $\phi : E(X,T) \to E(Y,S)$ such that $\pi(px) = \phi(p)\pi(x)$, $x \in X, p \in E(X,T)$. Moreover, for any filter $\mathcal{F}$, $\phi(H(X,\mathcal{F})) = H(Y,\mathcal{F})$.*

Let (X,T) be a TDS and I be any nonempty set. Let X^I be the product space. And we define $T : X^I \to X^I$ by $T(x_i)_{i\in I} = (Tx_i)_{i\in I}$. In the case I is a finite set, denote $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ and $T^{(n)} = \underbrace{T \times T \times \cdots \times T}_{n \text{ times}}$. The following result is not difficult to check (similar to [9, Proposition 3.9.]).

Proposition 2.4. *Let (X,T) be a TDS and I be any nonempty set. Then there is isomorphism $\psi : E(X,T) \cong E(X^I,T)$. Moreover, for any filter $\mathcal{F}$, $\psi : H(X,\mathcal{F}) \cong H(X^I,\mathcal{F})$.*

For a semigroup the element u with $u^2 = u$ is called *idempotent.* Ellis-Namakura Theorem says that for any Ellis semigroup E the set $Id(E)$ of idempotents of E is not empty [9]. A non-empty subset $I \subset E$ is a *left ideal* (resp. *right ideal*) if it $EI \subseteq I$ (resp. $IE \subseteq I$). I is said to be an *ideal* if it is both left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal of E. Obviously every left ideal is a semigroup and every left ideal contains some minimal ideal.

Let (X,T) be a TDS. Then (X^X,T) is a system and $(E(X,T),T)$ is its subsystem. A subset $I \subseteq E(X,T)$ is closed left ideal of $E(X,T)$ iff (I,T) is a subsystem of $(E(X,T),T)$. And I is minimal left ideal of $E(X,T)$ iff (I,T) is a minimal [4, 9].

Proposition 2.5. *If $\mathcal{F}$ is a filter, then $h(\mathcal{F})$ and $H(\mathcal{F})$ are closed nonempty sets. And if in addition $\mathcal{F}$ is thick (equivalently, it is translation −invariant), then $h(\mathcal{F})$ (resp. $H(\mathcal{F})$) is a closed left ideal of $\beta\mathbb{Z}_+$ (resp. $E(X,T)$).*

Proof. Recall the addition in $\beta\mathbb{Z}_+$ is

$$p+q=\{A\subseteq\mathbb{Z}_+:\{A-n\in q\}\in p\},$$

where $p,q\in\beta\mathbb{Z}_+$. For each $i\in\mathbb{Z}_+$ we denote by i^* the principal ultrafilter $\{A\subseteq\mathbb{Z}_+:i\in A\}$.

Now we show that $i^*+p\in h(\mathcal{F})$ for any $i\in\mathbb{Z}_+$ and $p\in h(\mathcal{F})$. First it is easy to check that $i^*+p=\{A\subseteq\mathbb{Z}_+:A-i\in p\}$. By the definition of $h(\mathcal{F})$, $p\in h(\mathcal{F})$ iff $\mathcal{F}\subseteq p$. Since $\mathcal{F}$ is translation −invariant, for any $A\in\mathcal{F}$ we have $A-i\in\mathcal{F}\subseteq p$. Hence $\mathcal{F}\subseteq i^*+p$, i.e. $i^*+p\in h(\mathcal{F})$ and $\mathbb{Z}_++h(\mathcal{F})\subseteq h(\mathcal{F})$. Since the operation $+$ is right-continuous operation on $\beta\mathbb{Z}_+$ and $h(\mathcal{F})$ is closed, $\beta\mathbb{Z}_++h(\mathcal{F})\subseteq h(\mathcal{F})$.

The conclusion concerning $H(\mathcal{F})$ follows from $H(\mathcal{F})=\Phi^{\#}(h(\mathcal{F}))$. □

We say that x *$\mathcal{F}$-adheres to* B if for any neighborhood U of B, $N(x,U)\in\mathcal{F}$ [2]. It is easy to see if B is closed then x $\mathcal{F}$ adheres to B iff for any $F\in k\mathcal{F}$, $\overline{T^F(x)}\cap B\neq\emptyset$. And x $\mathcal{F}$-adheres to y iff $y\in\omega_{\mathcal{F}}T(x)$.

Lemma 2.6. *Let (X,T) be a TDS , $B\subseteq X$ be closed and $\mathcal{F}$ be a proper family. Then*

(1) *If x $\mathcal{F}$-adheres to B, then $\omega_{k\mathcal{F}}T(x)\subseteq B$.*
(2) *If $\mathcal{F}$ is a filter, then x $k\mathcal{F}$-adheres to B iff $\omega_{k\mathcal{F}}T(x)\cap B\neq\emptyset$. And x $\mathcal{F}$-adheres to B iff $\omega_{k\mathcal{F}}T(x)\subset B$.*
(3) *A point x does not $\mathcal{F}$-adhere to B iff there is some closed B' separated from B such that x $k\mathcal{F}$ adheres to B'. Equivalently, x $\mathcal{F}$- adheres to B iff for any closed set B' such that x $k\mathcal{F}$-adheres to B' we have $B\cap B'\neq\emptyset$.*

Proof. (1). If not, then there is some $y\in\omega_{k\mathcal{F}}T(x)\setminus B$. Since B is closed, there are disjoint opene sets U,V which are neighborhoods of y and B respectively. But since $N(x,U)\in k\mathcal{F}$ and $N(x,V)\in\mathcal{F}$, we have $U\cap V\neq\emptyset$. A contradiction.

(2). Assume x $k\mathcal{F}$-adheres to B. Then $\overline{T^Fx}\cap B\neq\emptyset$ for each $F\in\mathcal{F}$. Since $\mathcal{F}$ is a filter, $\{\overline{T^Fx}\cap B:F\in\mathcal{F}\}$ has finite intersection property. Hence $\omega_{k\mathcal{F}}T(x)\cap B=\bigcap_{F\in\mathcal{F}}\overline{T^Fx}\cap B\neq\emptyset$. The converse is easy.

Now assume $\omega_{k\mathcal{F}}T(x)\subset B$. Let U be any neighborhood of B. Then $\bigcap_{F\in\mathcal{F}}\overline{T^Fx}\subseteq B\subseteq U$. As $\mathcal{F}$ is a filter, we have some $F\in\mathcal{F}$ such that $\overline{T^Fx}\subseteq U$. Hence x $\mathcal{F}$-adheres to B. The converse follows from (1).

(3) If x does not $\mathcal{F}$-adhere to B, then there is some $F\in k\mathcal{F}$ such that $\overline{T^Fx}\cap B=\emptyset$. It is easy to see that x $k\mathcal{F}$- adheres to $\overline{T^Fx}$. Conversely, suppose B' is closed and separated from B such that x $k\mathcal{F}$-adheres to B'. Let U,V be the neighborhoods of B,B' and $U\cap V=\emptyset$. Hence $N(x,U)\cap N(x,V)=\emptyset$. Since $N(x,V)\in k\mathcal{F}$, $N(x,U)\notin\mathcal{F}$. That is, x does not $\mathcal{F}$-adhere to B. □

A point $x \in X$ is said to be *$\mathcal{F}$-recurrent* if $x \in \omega_{\mathcal{F}}T(x)$. A point is said to be an *$\mathcal{F}$-transitive point* if $\omega_{\mathcal{F}}T(x) = X$. We denote the set of all $\mathcal{F}$-transitive points by $Trans_{\mathcal{F}}(X)$. For a system (X, T) if $Trans_{\mathcal{F}}(X) \neq \emptyset$, then (X, T) is said to *$\mathcal{F}$-transitive.* When $\mathcal{F} = \mathcal{B}$ we omit $\mathcal{F}$. Let $A \subseteq X$, the closure of the union of minimal subsets of A is called the *mincenter of A*. We can find the following results in [2], but the proofs we offer is different.

Proposition 2.7. *Let (X, T) be TDS, $x \in X$ and $B \subseteq X$ be closed.*

(1) *x $\mathcal{B}$-adheres to B iff $\omega T(x) \cap B \neq \emptyset$.*
(2) *x $k\mathcal{B}$-adheres to B iff $\omega T(x) \subseteq B$.*
(3) *x $\tau\mathcal{B}$-adheres to B iff B contains some invariant subset of $\omega T(x)$ iff B contains some minimal subset of $\omega T(x)$.*
(4) *x $k\tau\mathcal{B}$-adheres to B iff B intersect any minimal subset of $\omega T(x)$.*
(5) *x $\tau k\tau\mathcal{B}$-adheres to B iff B contains the mincenter of $\omega T(x)$.*
(6) *x $k\tau k\tau\mathcal{B}$-adheres to B iff B intersect the mincenter of $\omega T(x)$.*

Proof. (1) and (2) follow from Lemma 2.6-(2).

(3) Suppose x $\tau\mathcal{B}$-adheres to B. Then there is a sequence $\{a_n\} \subseteq \mathbb{Z}_+$ such that $T^{[a_n, a_n+n]}x \subset B_{\frac{1}{n}}$, where $B_\epsilon = \{y : d(y, B) < \epsilon\}$. Let $z = \lim_{n\to\infty} T^{a_n}x$ (take subsequence if necessary). Then $\overline{Orb(z, T)} \subseteq B$ and $\overline{Orb(z, T)}$ is invariant. The converse is easy.

(4) It follows from Lemma 2.6-(3).

(6) We show for any piecewise syndetic set F, there is some minimal point in $\overline{T^F x}$. Let $K = \overline{T^F x}$. Since F is piecewise syndetic, there is syndetic set F' such that for any $n \in \mathbb{N}$ there is some $a_n \in \mathbb{N}$ such that $a_n + (F' \cap [0, n]) \subset F$. Let r be the minimal number of F'. Then $T^{a_n+r} \in K$. Let $z = \lim_{n\to\infty} T^{a_n+r}x$ (take subsequence if necessary). Then $T^{F'-r}z \subseteq K$.

Let M be a bound on the gaps of F'. Then

$$\overline{Orb(z)} = \overline{T^{\mathbb{Z}_+}z} \subseteq \bigcup_{i=0}^{M} T^i K.$$

Let Y be a minimal subset of $\overline{Orb(z)}$. We now show $Y \cap K \neq \emptyset$. Let $y \in Y$ and $n_i \to \infty$ such that $T^{n_i}z \to y$. We can assume there is some $m \in [0, M]$ such that $T^{n_i+m}z \in K$. Hence $T^m y = \lim T^{n_i+m}z \in K$. So $T^m y \in Y \cap K$.

By this fact it is easy to see if x $k\tau k\tau\mathcal{B}$-adheres to B, then B intersects the mincenter of $\omega T(x)$. Now we show the converse. Assume that x does not $k\tau k\tau\mathcal{B}$-adhere to B. By Lemma 2.6-(3) there is some closed set B' such that $B \cap B' = \emptyset$ and x $\tau k\tau\mathcal{B}$-adheres to B'. Let U' and U be the disjoint neighborhoods of B' and B. then $F = N(x, U') \in \tau k\tau\mathcal{B}$. By the fact we proved above there is a minimal point in $\overline{T^F x}$. But $\overline{T^F x} \cap B = \emptyset$. This contradicts the fact that B intersect the mincenter of $\omega T(x)$.

(5) It follows from (6) and Lemma 2.6-(3). □

By Proposition 2.7, $\omega_{k\tau\mathcal{B}}T(x) = \emptyset$ unless $\omega T(x)$ contains a unique minimal subset M, in which case $\omega_{k\tau\mathcal{B}}T(x) = M$. And $\omega_{k\tau k\tau\mathcal{B}}T(x)$ is the mincenter of $\omega T(x)$.

Corollary 2.8. *Let (X,T) be a TDS and $x \in X$. Then*
(1) *x is $\tau\mathcal{B}$-recurrent iff x is a fixed point.*
(2) *x is $k\tau\mathcal{B}$-recurrent iff x is minimal point.*
(3) *x is $k\tau k\tau\mathcal{B}$-recurrent iff x is recurrent and the minimal points of $\omega T(x)$ is dense in $\omega T(x)$.*

3. Proximity relation

In this section for a family $\mathcal{F}$ the notion of $\mathcal{F}$-proximal relation is introduced. Basic properties of $\mathcal{F}$-proximal relation are discussed, and a new proof when the proximal relation is an equivalence one is presented. First we start with the definition of $\mathcal{F}$-proximal relation.

Definition 3.1. *Let (X,T) be a TDS and $x, y \in X$.*
(1) *Let $S \subset \mathbb{Z}_+$. (x,y) is said to be S-*proximal *if $\liminf_{S\ni n\to\infty} d(T^n x, T^n y) = 0$. (x,y) is said to be S-*asymptotic *if $\lim_{S\ni n\to\infty} d(T^n x, T^n y) = 0$. (x,y) is said to be S-*distal *if it is not S-proximal. We denote the set of all S-proximal pairs (resp. S-asymptotic pairs, S-distal pairs) by $P_S(X,T)$ (resp. $A_S(X,T), D_S(X,T)$).*
(2) *Let $\mathcal{F}$ be a family. $(x,y) \in X^2$ is called $\mathcal{F}$-*proximal *if (x,y) $\mathcal{F}$-adheres to Δ_X, i.e. for every $\epsilon > 0$, we have $N((x,y),\Delta_\epsilon) \in \mathcal{F}$, where $\Delta_\epsilon = \{(x,y) \in X^2 : d(x,y) < \epsilon\}$. We denote the set of all $\mathcal{F}$-proximal pairs by $P_{\mathcal{F}}(X,T)$ or $P_{\mathcal{F}}$.*

Remark 3.2. (1) *It is easy to see that (x,y) is $\mathcal{F}$-proximal iff for every $F \in k\mathcal{F}$, $\overline{(T\times T)^F(x,y)} \cap \Delta_X \neq \emptyset$ iff $(x,y) \in P_F(X,T)$ for any $F \in k\mathcal{F}$. And*

$$P_{\mathcal{F}}(X,T) = P_{\mathcal{F}} = \bigcap_{\epsilon>0} \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} (T\times T)^{-n}\Delta_\epsilon = \bigcap_{\epsilon>0} \bigcap_{F\in k\mathcal{F}} (T\times T)^{-F}\Delta_\epsilon.$$

From this we can see $P = P_{\mathcal{B}}$ is a G_δ set.
(2) *By Proposition 2.1 and Lemma 2.6, if $\mathcal{F}$ is a filter then (x,y) is $k\mathcal{F}$-proximal iff there is $p \in H(\mathcal{F})$ such that $px = py$. And (x,y) is $\mathcal{F}$-proximal iff for every $p \in H(\mathcal{F})$, $px = py$. Take $\mathcal{F} = k\mathcal{B}$, then (x,y) is proximal iff (x,y) is $\mathcal{B}$ proximal iff there is $p \in H(k\mathcal{B})$ such that $px = py$. (x,y) is asymptotic iff (x,y) is $k\mathcal{B}$ proximal iff $px = py$ for every $p \in H(k\mathcal{B})$.*

Proposition 3.3. *Let (X,T) be TDS and $\mathcal{F}$ be a thick filter. Then the following are equivalent:*
(1) *$(x,y) \in P_{k\mathcal{F}}$.*
(2) *$px = py$ for some $p \in H(\mathcal{F})$.*
(3) *There is some minimal left ideal I contained in $H(\mathcal{F})$ such that $px = py$ for each $p \in I$.*
If in addition (X,T) is minimal then $(1)-(4)$ are equivalent
(4) *There is some minimal idempotent u of $H(\mathcal{F})$ such that $y = ux$.*

Proof. (1) $\Rightarrow$ (2): Assume $(x,y) \in P_{k\mathcal{F}}$. Then $\omega_{k\mathcal{F}}(T,(x,y)) \cap \Delta_X \neq \emptyset$. So there is some $p \in H(\mathcal{F})$ such that $px = py$.

(2) $\Rightarrow$ (3): Assume $px = py$ for some $p \in H(\mathcal{F})$. Then $L = \{p \in H(\mathcal{F}) : px = py\}$ is nonempty closed left ideal as $H(\mathcal{F})$ is a left ideal. Hence by Zorn's Lemma there is some minimal left ideal I contained in L. Hence $px = py$ for each $p \in I$.

(3) $\Rightarrow$ (1): It follows easily by Remark 3.2.

Now assume (X, T) is minimal. Then

(3) $\Rightarrow$ (4): Since (X, T) is minimal, $Iy = X$. Hence $S = \{p \in I : py = y\}$ is a nonempty semigroup, and by Ellis-Namakura Theorem there is an idempotent $u \in S \subset I$. As I is a minimal left ideal, u is a minimal idempotent. Thus, we have $y = uy = ux$.

(4) $\Rightarrow$ (1): If $y = ux$, then $u(x, y) = u(x, ux) = (ux, ux) \in \Delta$. By $u \in H(\mathcal{F})$ and Remark 3.2, (x, y) is $k\mathcal{F}$-proximal. □

Now we discuss the condition when $P_{\mathcal{F}}$ is an equivalence relation. First we have

Proposition 3.4. *Let (X, T) be a TDS and $\mathcal{F}$ be a full family.*

(1) $P_{\mathcal{F}} = P_{\tau\mathcal{F}}$.

(2) If $\tau\mathcal{F}$ is a filter, then $P_{\mathcal{F}}$ is an equivalence relation.

(3) If $\tau\mathcal{F}$ is a translation invariant filter, then $P_{\mathcal{F}}$ is a $T \times T$-invariant equivalence relation.

Proof. (1) It remains to show $P_{\mathcal{F}} \subseteq P_{\tau\mathcal{F}}$. Let $(x, y) \in P_{\mathcal{F}}$ and $A = N((x, y), \Delta_\epsilon)$. For some fixed $N \in \mathbb{N}$ and any $\epsilon > 0$ there is some $\delta > 0$ such that if $d(x', y') < \delta$ then $d(T^n x', T^n y') < \epsilon, 1 \le n \le N$. Set $A' = N((x, y), \Delta_\delta)$. By definition $A' \in \mathcal{F}$ and $A' \subseteq \{t : [t, t+N] \subset A\}$. Hence $\{t : [t, t+N] \subset A\} \in \mathcal{F}$. This means $A \in \tau\mathcal{F}$. So we have $(x, y) \in P_{\tau\mathcal{F}}$.

(2) and (3) are easy by (1). □

Remark: In [8] the author showed that $P_{k\tau\mathcal{B}}$ is an equivalence relation. Here by the fact that $\tau k\tau\mathcal{B}$ is a filter we get a more straightforward proof.

Let $\{(X_i, T_i)\}_{i\in I}$ be a family of TDS. Let $\eta : (\Pi_{i\in I} X_i)^2 \to \Pi_{i\in I} X_i^2$ be the map such that $((x_i)_{i\in I}, (y_i)_{i\in I}) \mapsto (x_i, y_i)_{i\in I}$. Then η is a homeomorphism and maps the relation of $(\Pi_{i\in I} X_i)^2$ into the relation of $\Pi_{i\in I} X_i^2$, i.e. $\eta R(\Pi_{i\in I} X_i, T) \subseteq \Pi_{i\in I} R(X_i, T)$, where R is some relation.

Let $\{(X_i, T_i)\}_{i\in I}$ be a family of TDS, then the product space $\Pi_{i\in I} X_i$ is Hausdoff and compact (by Tychonoff Theorem) but generally not metric. Hence we have to generalize the proximity to uniform space. Let (X, T) be a uniform space and $\mathcal{F}$ be a family, $(x, y) \in X^2$ is $\mathcal{F}$ proximal if for any index α, $N((x, y), \alpha) \in \mathcal{F}$. By the definition of product topology it is easy to prove the following proposition.

Proposition 3.5. *Let $\{(X_i, T_i)\}_{i\in I}$ be a family of TDS and $\mathcal{F}$ be a family. Then $\eta P_{\mathcal{F}}(\Pi_{i\in I} X_i, T) \subseteq \Pi_{i\in I} P_{\mathcal{F}}(X_i, T)$. If $\mathcal{F}$ is a filter, then $\eta P_{\mathcal{F}}(\Pi_{i\in I} X_i, T) = \Pi_{i\in I} P_{\mathcal{F}}(X_i, T)$.*

The following results appeared in [3, 8, 17]. Here we offer a different and more straightforward proof.

Theorem 3.6. *Let (X, T) be a TDS. Then the following conditions are equivalent*

(1) *P is an equivalence relation;*

(2) *$P = P_{k\tau\mathcal{B}}$;*

(3) *Every orbit closure in* $(X \times X, T \times T)$ *contains precisely one minimal set.*
(4) *Every orbit closure in* (X^I, T) *contains precisely one minimal set, where* I *is any nonempty set.*
(5) *There is only one minimal right ideal in the enveloping semigroup* $E(X,T)$.

Proof. (1) $\Rightarrow$ (2): If $P \neq P_{k\tau\mathcal{B}}$, then there is some $(x,y) \in P \setminus P_{k\tau\mathcal{B}}$, i.e. there is some $F \in \tau\mathcal{B}$ such that $\overline{T^F(x,y)} \cap \Delta = \emptyset$. Let M be an invariant set of $\overline{T^F(x,y)}$. (As F is thick, let $\{n_i\}_{i=i}^{\infty} \subseteq F$ with $[n_i, n_i+i] \subseteq F$ for any $i \in \mathbb{N}$. Let $z = \lim T^{n_i}(x,y)$. Then $M = \overline{Orb(z,T)}$ is an invariant subset of $\overline{T^F(x,y)}$.)

Since (x,y) is proximal to M, there is some $(x',y') \in M$ such that (x,y) and (x',y') are proximal. Especially, $(x,x'),(y,y') \in P$. Together with $(x,y) \in P$ we have $(x',y') \in P$ as P is transitive. Hence $\overline{Orb((x',y'),T)} \cap \Delta \neq \emptyset$. This is a contradiction, since $\overline{Orb((x',y'),T)} \subseteq M$ and $M \cap \Delta = \emptyset$.

(2) $\Rightarrow$ (1): Since $P = P_{k\tau\mathcal{B}} = P_{\tau k\tau\mathcal{B}}$ and $\tau k\tau\mathcal{B}$ is a filter, P is transitive and hence is an equivalence relation.

(2) $\Rightarrow$ (4): Assume there is some $z \in (X^I,T)$ such that there are two disjoint minimal sets M_1, M_2 in $\overline{Orb(z)}$. Let $z_i \in M_i$ such that $(z,z_i) \in P(X^I,T)$, $i=1,2$. Hence $(z_1,z_2) \in P(X^I,T)$ by condition (2) and Proposition 3.5. Then $M_1 \cap M_2 \neq \emptyset$. A contradiction.

(4) $\Rightarrow$ (5): Let $I = X$ and by $E(X,T) = \overline{Orb(id,T)}$ the result follows.

(5) $\Rightarrow$ (3): For any $z \in X^2$, $E(X,T)z = \overline{Orb(z,T^{(2)})}$. As there is only one minimal right ideal in the enveloping semigroup, there is only one minimal set in $\overline{Orb(z,T^{(2)})}$.

(3) $\Rightarrow$ (2): If $P \neq P_{k\tau\mathcal{B}}$, then there is some $z \in P \setminus P_{k\tau\mathcal{B}}$, i.e. there is some $F \in \tau\mathcal{B}$ such that $\overline{T^F z} \cap \Delta = \emptyset$. Let M be an invariant set of $\overline{T^F z}$. Since $z \in P$, $\overline{Orb(z,T^{(2)})} \cap \Delta \neq \emptyset$. As $\overline{Orb(z,T^{(2)})} \cap \Delta$ is a closed invariant subset of $\overline{Orb(z,T^{(2)})}$, it contains some minimal set M'. Thus we get two distinct minimal subsets in $\overline{Orb(z,T^{(2)})}$. This contradicts (3). $\square$

Now we give a more general statement of the above result. Since the proof is similar to the above one and we omit it.

Theorem 3.7. *Let* (X,T) *be TDS and* $\mathcal{F}$ *be an invariant filter. Then the following conditions are equivalent*

(1) $P_{k\mathcal{F}}$ *is an equivalence relation;*
(2) $P_{k\mathcal{F}} = P_{k\tau k\mathcal{F}}$;
(3) *There is only one minimal set in* $\omega_{k\mathcal{F}}T(z)$ *for every point* $z \in (X \times X, T \times T)$.
(4) *There is only one minimal set in* $\omega_{k\mathcal{F}}T(z)$ *for every point* $z \in (X^I, T)$, *where* I *is any nonempty set.*
(5) *There is only one minimal right ideal in* $H(\mathcal{F})$.

4. Distality concepts

This section is devoted to discuss the distality concepts via families. Namely, for a family $\mathcal{F}$, the notion of $\mathcal{F}$-Li-Yorke pairs and $\mathcal{F}$-almost distality are introduced, and $\mathcal{F}$-almost distality is characterized. Let us see first how the notions comes from the previous section.

It is easy to see that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $P_{\mathcal{F}_1} \subseteq P_{\mathcal{F}_2}$. Hence we have

$$\Delta = P_{\mathbb{Z}_+} \subseteq P_{k\mathcal{B}} = Asymp \subseteq \cdots \subseteq P_{\mathcal{F}} \subseteq \cdots \subseteq P_{\mathcal{B}} = P$$

We have the following observations.

1. When $P = P_{\mathbb{Z}_+}$, (X,T) is distal and $H(k\mathcal{B})$ is a group;

2. When $P = P_{k\mathcal{B}}$, (X,T) is almost distal and $H(k\mathcal{B})$ itself is the minimal left ideal;

3. When $P = P_{k\tau\mathcal{B}} = P_{\tau k\tau\mathcal{B}}$, P is an equivalence relation and $H(k\mathcal{B})$ has only one minimal left ideal.

It is naturel to ask the following questions:

(1) When $P = P_{\mathbb{Z}_+}$, (X,T) is distal. Then what is the case if $P_{\mathcal{F}} = \Delta = P_{\mathbb{Z}_+}$?

(2) When $P_{\mathcal{B}} = P = P_{k\mathcal{B}}$, (X,T) is almost distal. In other words this means (X,T) has no Li-Yorke pairs. Then what is the case if $P_{\mathcal{F}} = P_{k\mathcal{F}}$?

To answer (1) we need the notion of $\mathcal{F}$-distality which was introduced in [2]. To answer (2) we introduce the notion of $\mathcal{F}$-Li-Yorke pairs and $\mathcal{F}$-almost distality.

Definition 4.1. *(X,T) is called* $\mathcal{F}$-distal *if $P_{k\mathcal{F}} = \Delta_X$.*

It is easy to see that (X,T) is $\mathcal{F}$-distal iff for any $(x,y) \in X \setminus \Delta_X$ there is some $F \in \mathcal{F}$ such that $(x,y) \in D_F$ (i.e. $(x,y) \notin P_F$). $k\mathcal{B}$-distal is the usual concept of distality. (X,T) is $\mathcal{B}$- distal iff it contains no proper asymptotic pair.

Proposition 4.2. *Let $\mathcal{F}$ be a full family. Then any $\mathcal{F}$-distal system has zero topological entropy.*

Proof. By [6] $Asym(X,T)\setminus\Delta \neq \emptyset$ for any positive entropy systems. By the definition of $\mathcal{F}$-distality, $Asym(X,T) = \Delta$ and the result follows. □

$\mathcal{F}$-distality is studied in [2], and the following nice result was proved.

Proposition 4.3. [2] *Let (X,T) be a TDS and $\mathcal{F}$-a filter such that $H(\mathcal{F})$ is a semigroup of $E(X,T)$. Then (X,T) is $\mathcal{F}$-distal iff $H(\mathcal{F})$ is a group of bijections of X.*

Now we discuss the notion of $\mathcal{F}$-almost distality.

Definition 4.4. *Let (X,T) be a TDS and $\mathcal{F}$ be a family.*

(1) *(x,y) is said to be* $\mathcal{F}$-Li-Yorke pair *if (x,y) is $k\mathcal{F}$- proximal but not $\mathcal{F}$-proximal, i.e. $(x,y) \in P_{k\mathcal{F}} \setminus P_{\mathcal{F}}$. We denote the set of all $\mathcal{F}$-Li-Yorke pairs by $LY_{\mathcal{F}}$.*

(2) *(X,T) is said to be* $\mathcal{F}$-almost distal *if (X,T) has no $\mathcal{F}$- Li-Yorke pair.*

Remark 4.5. *1. $(x,y) \in LY_{\mathcal{F}}$ iff for any $F \in \mathcal{F}$, $(x,y) \in P_F$ and there is $F \in k\mathcal{F}$ such that $(x,y) \notin P_F$. If in addition $\mathcal{F}$ is a filter then by Remark 3.2 $(x,y) \in LY_{\mathcal{F}}$ iff there are $p_1, p_2 \in H(\mathcal{F})$ such that $p_1x = p_1y$ and $p_2x \neq p_2y$.*

2. Let (X,T) be a TDS and $\mathcal{F}$ be a filter. Then (X,T) is $\mathcal{F}$-almost distal iff $P_{\mathcal{F}} = P_{k\mathcal{F}}$ iff if $(x,y) \in X^2$ is $k\mathcal{F}$-proximal then for any $p \in H(\mathcal{F})$, $px = py$.

Theorem 4.6. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter. Then (X,T) is $\mathcal{F}$-almost distal iff $(H(\mathcal{F}),T)$ is minimal.*

Proof. Let (X,T) be $\mathcal{F}$-almost distal. As $(H(\mathcal{F}),T)$ is a subsystem of $(H(k\mathcal{B}),T)$ by Proposition 2.5, there is a minimal subset $I \subset H(\mathcal{F})$. Set $u \in I$ be an idempotent. Then for any $x \in X$, by $ux = u(ux)$ and $u \in H(\mathcal{F})$ (x, ux) is $k\mathcal{F}$-proximal. As (X,T) is $\mathcal{F}$-almost distal, $px = pux$ for any $p \in H(\mathcal{F})$. Hence we have $px = pux$ for any $x \in X$. That is, $p = pu \in H(\mathcal{F})I \subset H(k\mathcal{B})I \subset I$. Thus $H(\mathcal{F}) = I$, i.e. $(H(\mathcal{F}),T)$ is minimal.

Now show the converse. Suppose (x,y) is an $\mathcal{F}$-Li-Yorke pair. Then there is $p \in H(\mathcal{F})$ such that $px = py$. Let $I = \{p \in H(\mathcal{F}) : px = py\}$. I is a nonempty closed left ideal of $H(k\mathcal{B})$. As (x,y) is not $\mathcal{F}$-proximal, there is $q \in H(\mathcal{F})$ such that $qx \neq qy$. Hence $q \notin I$ and $H(\mathcal{F}) \neq I$. Thus, it follows that $(H(\mathcal{F}),T)$ is not minimal. □

Corollary 4.7. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter. If (X,T) is $\mathcal{F}$-almost distal, then any factor of (X,T) is $\mathcal{F}$-almost distal. Also for any nonempty set I, the product system (X^I,T) is $\mathcal{F}$-almost distal.*

Proof. It follows from Theorem 4.6, Propositions 2.3 and 2.4. □

Corollary 4.8. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter. Then*

(1) *If (X,T) is $\mathcal{F}$-almost distal, then for every $x \in X$, $\omega_{k\mathcal{F}}(T,x)$ is minimal.*
(2) *(X,T) is $\mathcal{F}$-almost distal iff for every $(x,y) \in X^2$, $\omega_{k\mathcal{F}}(T \times T,(x,y))$ is minimal.*

Proof. (1) Since $H(\mathcal{F})$ is a minimal left ideal, $\omega_{k\mathcal{F}}(T,x) = H(\mathcal{F})x$ is minimal.

(2) As (X,T) is $\mathcal{F}$-almost distal, and hence $(X \times X, T \times T)$ is $\mathcal{F}$-almost distal. By (1) for every $(x,y) \in X^2$ $\omega_{k\mathcal{F}}(T \times T,(x,y))$ is minimal. Now we show the converse. Suppose (x,y) is $k\mathcal{F}$-proximal, then $\omega_{k\mathcal{F}}(T \times T,(x,y)) \cap \Delta_X \neq \emptyset$. As $\omega_{k\mathcal{F}}(T \times T,(x,y))$ is minimal, $\omega_{k\mathcal{F}}(T \times T,(x,y)) \subset \Delta_X$. That is, (x,y) is $\mathcal{F}$-proximal and by the definition it is not an $\mathcal{F}$-Li-Yorke pair. Hence (X,T) is $\mathcal{F}$-almost distal. □

To end the section we give a negative answer to some problem in [7]. By a $\mathbb{Z}$-system (X,T) we mean $T : X \to X$ is a homeomorphism and the action group is $\mathbb{Z}$. In this case, all notions are similar to the the ones used before. For example, a pair $(x,y) \in X^2$ is proximal if $\liminf_{|n|\to\infty} d(T^n x, T^n y) = 0$ and is asymptotic if $\lim_{|n|\to\infty} d(T^n x, T^n y) = 0$. We call the $\mathbb{Z}$-system (X,T) *almost distal* if every proximal pair $(x,y) \in X$ is asymptotic. In [7] the authors asked whether or not any transitive almost distal $\mathbb{Z}$-system is an asymptotic extension of a transitive distal system. That is, whether $P = P_{\mathcal{B}} = P_{k\mathcal{B}} = Asym(X,T)$ implies that P is a closed equivalence relation. Since there is a minimal $\mathbb{Z}$-system (X,T) such that $P(X,T) = Asym(X,T)$ is an equivalence but not a closed relation [15], and hence it is not an asymptotic extension of a distal system. Thus, the answer to the question in [7] is negative. It indicates that the condition that $P_{\mathcal{F}}$ is closed is not easy to be satisfied, and the structure of an almost distal system is not as simple as we have thought before.

5. Recurrence and $\mathcal{F}$-semi-distality

In this section we will give some characterizations of $\mathcal{F}$-semi-distality and PI-flows. Let (X, T) be a TDS and $\mathcal{F}$ be a family. Recall that $x \in X$ is said to be $\mathcal{F}$-recurrent if $x \in \omega_{\mathcal{F}}(T, x)$, i.e. for every neighborhood U of x, $N(x, U) \in \mathcal{F}$.

Definition 5.1. *Let (X, T) be a TDS and $\mathcal{F}$ be a family.*

(1) $(x, y) \in X^2 \setminus \Delta_X$ *is a* strong $\mathcal{F}$-Li-Yorke pair *if (x, y) is $k\mathcal{F}$- proximal and $k\mathcal{F}$-recurrent, i.e. for any $F \in \mathcal{F}$, $(x, y) \in P_F$ and $N((x, y), U) \cap F \neq \emptyset$, where U is any neighborhood of (x, y).*

(2) *Call (X, T)* $\mathcal{F}$-semi-distal *if (X, T) has no strong $\mathcal{F}$-Li-Yorke pair.*

Remark 5.2. (1) *Let $\mathcal{F}$ be a filter. Then (x, y) is a strong $\mathcal{F}$-Li-Yorke pair iff there are $p_1, p_2 \in H(\mathcal{F})$ such that $p_1 x = p_1 y$ and $p_2(x, y) = (x, y)$.*

(2) *Any strong $\mathcal{F}$-Li-Yorke pair is an $\mathcal{F}$-Li-Yorke pair. And hence any $\mathcal{F}$-almost distal system is $\mathcal{F}$-semi-distal.*

To give some equivalence conditions of $\mathcal{F}$-semi-distality, we need some basic results on the idempotents of Ellis semigroup.

Ellis-Namakura Theorem says that for any Ellis semigroup E the set $Id(E)$ of idempotents of E is not empty. We can introduce a quasi-order (a reflexive, transitive relation) $<_R$ on the set $Id(E)$ by defining $v <_R u$ iff $uv = v$. If $v <_R u$ and $u <_R v$ we say that u and v are *equivalent* and write $u \sim_R v$. Similarly, we define $<_L$ and $\sim_L$. An idempotent $u \in Id(E)$ is *minimal* if $v \in Id(E)$ and $v <_R u$ implies $u <_R v$. The following lemma is well-known [1, 11], and for completeness we include a proof.

Lemma 5.3. (1) *Let L be a left ideal of Ellis semigroup S and $u \in Id(S)$. Then there is some idempotent v in Lu such that $v <_R u$ and $v <_L u$.*

(2) *An idempotent is minimal iff it is contained in some minimal left ideal.*

Proof. (1) First note that Lu is also a left ideal. By Ellis-Namakura Theorem there is some $w \in Id(Lu)$. Let $v = uw$. Then $v \in uLu \subseteq Lu$ and $v^2 = uwuw = uww = uw = v$. And we have $vu = v$ and $uv = v$, i.e. $v <_R u$ and $v <_L u$.

(2) Let u be a minimal idempotent and L be a left minimal ideal. Then by (1) there is some idempotent v in Lu such $v <_R u$. Since u is minimal, we have $u <_R v$. Thus $u = vu \in Lu$. As Lu is a minimal ideal, the result follows.

Conversely, let L be a minimal left ideal and $u \in Id(L)$. Let $v \in S$ be any idempotent such that $v <_R u$. Then by $(vu)(vu) = vvu = vu$, vu is an idempotent of L. As L is minimal, $L(vu) = L$. Then there is some $p \in L$ such $pvu = u$. Thus $vu = (uv)u = u(vu) = p(vu)(vu) = pvu = u$, i.e. $u <_R v$. That is, u is minimal. □

By this lemma we have the following readily.

Corollary 5.4. *Let L be a left ideal of Ellis semigroup S and $u \in Id(L)$. Then there is some minimal idempotent v in L such that $v <_R u$ and $v <_L u$.*

The following proposition is needed for the proof of the next theorem.

Proposition 5.5. *Let (X,T) be a TDS, $\mathcal{F}$ be a thick filter and $x \in X$ be a $k\mathcal{F}$-recurrent point. Then there is some minimal point $y \in \omega_{k\mathcal{F}}T(x)$ such that $(x,y) \in P_{k\mathcal{F}}$ and it is a $k\mathcal{F}$-recurrent point of $(X^2, T\times T)$.*

Especially, for any recurrent point x, there is some minimal point y in the orbit closure of x which is proximal to x and (x,y) is a recurrent point of $(X^2, T\times T)$.

Proof. Since $x \in \omega_{k\mathcal{F}}T(x) = H(\mathcal{F})x$, there is some $u \in Id(H(\mathcal{F}))$ such that $ux = x$. By Corollary 5.4 there is a minimal idempotent $v \in H(\mathcal{F})$ with $vu = uv = v$. Let $y = vx$, then

$$u(x,y) = (ux, uvx) = (ux, vx) = (x,y),$$
$$v(x,y) = (vx, vvx) = (y,y).$$

Thus the statement follows. □

Theorem 5.6. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter. Then (X,T) is $\mathcal{F}$-semi-distal iff any idempotent of $H(\mathcal{F})$ is minimal.*

Proof. Assume that (X,T) is $\mathcal{F}$-semi-distal. Let $u \in H(\mathcal{F})$ is idempotent. By Corollary 5.4 there is some minimal idempotent $v \in H(\mathcal{F})$ such that $vu = uv = v$. Then for any $x \in X$

$$u(ux, vx) = (u^2x, uvx) = (ux, vx),$$
$$v(ux, vx) = (vux, v^2x) = (vx, vx).$$

That is, (ux, vx) is a strong $\mathcal{F}$-Li-Yorke pair. As (X,T) is $\mathcal{F}$- semi-distal, $ux = vx$. Since x is arbitrary, $u = v$. In particular u is minimal.

Now we show the converse. Let (x,y) be any strong $\mathcal{F}$-Li-Yorke pair. Then (x,y) is $k\mathcal{F}$-recurrent, and since any idempotent is minimal, it follows that (x,y) is a minimal point of $(X^2, T\times T)$. But (x,y) is also proximal. Hence $(x,y) \in \Delta$. That is, (X,T) is $\mathcal{F}$-semi-distal. □

Corollary 5.7. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter. If (X,T) is $\mathcal{F}$-semi-distal, then any factor of (X,T) is $\mathcal{F}$-semi-distal. Also for any nonempty set I, the product system (X^I, T) is $\mathcal{F}$-semi-distal.*

Proof. It follows from Theorem 5.6, Propositions 2.3 and 2.4. □

Corollary 5.8. *Let (X,T) be a TDS and $\mathcal{F}$ be a thick filter.*

(1) *If (X,T) is $\mathcal{F}$-semi-distal then every $k\mathcal{F}$-recurrent point is minimal.*
(2) *(X,T) is $\mathcal{F}$-semi-distal iff every $k\mathcal{F}$-recurrent point of X^2 is minimal.*

Proof. (1) Assume x is $k\mathcal{F}$-recurrent. Then $x \in \omega_{k\mathcal{F}}(T,x) = H(\mathcal{F})x$. As $H(\mathcal{F})$ is an ideal, there is an idempotent $u \in H(\mathcal{F})$ such that $ux = x$. By Theorem 5.6 u is minimal and hence x is minimal.

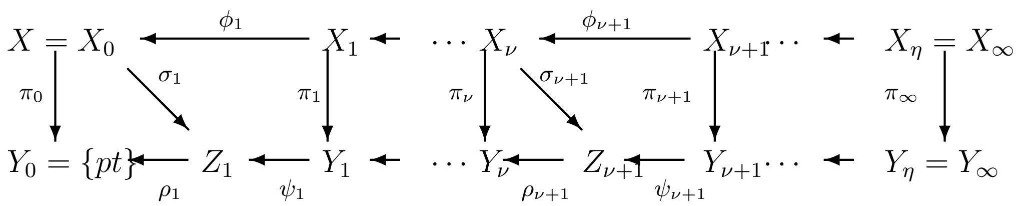
(2) The first part follows from (1). Now we suppose every $k\mathcal{F}$-recurrent point of $X \times X$ is minimal. If $(x,y) \in X\times X$ is $k\mathcal{F}$-proximal, then $\omega_{k\mathcal{F}}(T\times T, (x,y)) \cap \Delta_X \neq \emptyset$. Since $\omega_{k\mathcal{F}}(T\times T, (x,y))$ is minimal, $\omega_{k\mathcal{F}}(T\times T, (x,y)) \subset \Delta_X$. So $x = y$. That is, (X,T) is $\mathcal{F}$-semi-distal. □

Corollary 5.9. *Let $\mathcal{F}$ be a thick filter. If a TDS (X, T) is $\mathcal{F}$-semi-distal and $k\mathcal{F}$-transitive, then it is minimal. In particular, any transitive semi-distal system is minimal.*

Now we will shall how to interpret PI-flows as some $\mathcal{F}$-semi-distal ones. Given two $\mathbb{Z}$-systems (X, T) and (Y, S), a continuous map $\pi : X \to Y$ is called a *homomorphism* of systems (X, T) and (Y, S) if it is onto and $\pi T = S\pi$. We say (X, T) is an entension of (Y, S). If π is also injective then it is called an isomorphism. An extension $\pi : X \to Y$ is called *proximal* (resp. *distal*) if $\pi(x_1) = \pi(x_2)$ implies that x_1 and x_2 are proximal (resp. distal). It is called *equicontinuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\pi(x_1) = \pi(x_2)$ and $d(x_1, x_2) < \delta$ imply $d(T^n(x_1), T^n(x_2)) < \epsilon$ for any $n \in \mathbb{Z}$. An equicontinuous extension is also called an *isometric extension*. The extension π is *almost one to one* if there exists a dense G_δ set $X_0 \subset X$ such that $\pi^{-1}\pi(x) = \{x\}$ for any $x \in X_0$. Finally the extension is called *weak mixing* if the subsystem $R_\pi = \{(x_1, x_2) : \pi(x_1) = \pi(x_2)\}$ is toplogically transitive under $T \times T$.

The following theorem is the structure theorem for minimal systems [4][Theorem 14.30].

Theorem 5.10 (Structure theorem for minimal systems)**.** *Given a compact metric minimal system (X, T), there exists a countable ordinal η and a canonically defined commutative diagram of minimal systems (it is called* **PI-tower***):*



where for each $\nu \le \eta$, ρ_ν is equicontinuous, ϕ_ν and ψ_ν are proximal and π_∞ is open and weakly mixing. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits of $X_\lambda, Y_\lambda, \pi_\lambda$ etc. for $\lambda < \nu$.

(X, T) is said to be *strictly proximal isometric* or *strictly PI* if it has structure as (Y_∞, T) in PI-tower, i.e. it can be get from the trivial system by a (countable) transfinite succession of proximal and equicontinuous extensions. And (X, T) is said to be *proximal isometric* or *PI* if in PI-tower π_∞ is isomorphic, or equivalently it is the factor of a strictly PI system by a proximal extension.

If in the above definitions proximal is replaced by almost one to one then we get the notions of *strictly HPI system* and *HPI system*.

Theorem 5.11. [16] *Let (X, T) be a minimal dynamical system. Then*

(1) *X is PI iff it satisfies the following property: whenever W is a closed invariant subset of $X \times X$ which is topologically transitive and has a dense subset of minimal points, then W is minimal.*
(2) *X is HPI iff every transitive subsystem $Y \subseteq X \times X$ such that every projection $\pi_i : Y \to X, i = 1, 2$ is semi-open (i.e. the image of every nonempty open set has nonempty interior) is minimal.*

By this theorem we can get the following results readily.

Proposition 5.12. *Let (X, T) be a minimal system. If (X, T) is semi-distal, then it is an HPI-flow, i.e. pointed distal.*

Theorem 5.13. *Let (X, T) be minimal system. Then (X, T) is a PI-flow iff it is $\tau k \tau \mathcal{B}$-semi-distal.*

Proof. It follows from Theorem 5.11, Corollary 5.8 and Corollary 2.8. □

Let K be the smallest ideal of $\beta\mathbb{Z}_+$. It is well known that K is the union of all minimal left ideals and is also the union of all minimal right ideals. By Proposition 2.7 it is easy to get $h(\tau k \tau \mathcal{B}) = \overline{K}$. In [13] the authors showed the algebraic structure of $\overline{K} \setminus K$ is indeed very rich. For example, they showed there are 2^c idempotents in $\overline{K} \setminus K$, where c is the cardinality of the continuum. Now Let (X, T) be a TDS and M be the smallest ideal of $E(X, T)$. Then $H(\tau k \tau \mathcal{B}) = \overline{M}$. But in this case we don't know whether there are idempotents in $\overline{M} \setminus M$ or not. Surely when (X, T) is semi-distal there is no idempotent in $\overline{M} \setminus M$, as every idempotent of $\mathcal{H}(X)$ is minimal and hence in M. In general we have that the condition that there is no idempotent in $\overline{M} \setminus M$ is equivalent to PI. Equivalently, we have

Theorem 5.14. *Let (X, T) be minimal system. Then (X, T) is a PI-flow iff any idempotent of $H(\tau k \tau \mathcal{B})$ is minimal.*

Proof. It follows from Theorem 5.6 and Theorem 5.13. □

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