MIXING AND PROXIMAL CELLS ALONG SEQUENCES

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Abstract. A dynamical system (X,T) is $\mathcal{F}$-transitive if for each pair of open and non-empty subsets U and V of X, $N(U,V)=\{n\in\mathbb{Z}_+ : U\cap T^{-n}V\neq\emptyset\}\in\mathcal{F}$, where $\mathcal{F}$ is a collection of subsets of $\mathbb{Z}_+$ which is hereditary upward. (X,T) is $\mathcal{F}$-mixing if $(X\times X, T\times T)$ is $\mathcal{F}$-transitive. For a subset S of $\mathbb{Z}_+$, $(x,y)\in X\times X$ is S-proximal if $\liminf_{S\ni n\to+\infty} d(T^n(x),T^n(y))=0$ and the S-proximal cell $P_S(x)$ is the set of points which are S-proximal to $x\in X$. We show that if (X,T) is $\mathcal{F}$-mixing then for each $S\in k\mathcal{F}$ (the dual family of $\mathcal{F}$) and $x\in X$, $P_S(x)$ is a dense G_δ subset of X, and when (X,T) is minimal and $\mathcal{F}$ is a filter the reciprocal is true. Moreover, other conditions under which the reciprocal is true are obtained. Finally the structure of proximal cells for $\mathcal{F}$-mixing systems is discussed, and a new and simpler proof of the Xiong-Yang's theorem is presented.

§1 introduction

Throughout this paper a *topological dynamical system* (TDS for short) is a pair (X,T), where X is a nonvoid compact metric space with a metric d and T is a continuous surjective map from X to itself. Recall that (X,T) is *transitive* if for each pair of opene (i.e. nonempty and open) subsets U and V, $N(U,V)=\{n\in\mathbb{Z}_+ : U\cap T^{-n}V\neq\emptyset\}$ is non-empty. (X,T) is *(topologically) weakly mixing* if $(X\times X, T\times T)$ is transitive. Let $\omega(T,x)$ be the set of the limit points of orbit of x, $Orb(x,T)=\{x,T(x),T^2(x),\dots\}$. $x\in X$ is called a *transitive point* if $\omega(T,x)$ is dense in X. It is easy to see that if (X,T) is transitive then the set of all transitive points is a dense G_δ set of X (denoted by $Trans_T$). If $Trans_T=X$ then we say that (X,T) is *minimal.* Equivalently, (X,T) is minimal iff it contains no proper subsystems. It is well known that there is some minimal subsystem in any dynamical system (X,T), which is called a *minimal set* of X. Each point belonging to some minimal set of X is called a *minimal point.*

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Classically in topological dynamical systems pairs of points are considered from the asymptotic behavior of their trajectories. A pair $(x, y) \in X \times X$ is said to be *proximal* if $\liminf_{n\to+\infty} d(T^n(x), T^n(y)) = 0$ and the set of all proximal pairs is denoted by P. P is a reflexive, symmetric, $T-$invariant relation, but is not in general transitive or closed [Au1, Au2, Au3]. For $x \in X$ it is interesting and useful to consider the points which are proximal to x. That is, we are interested in the set $P(x) = \{y \in X : (x, y) \in P\}$, which is called the *proximal cell of x*.

The first and maybe the most important result concerning the proximal cell is that for any $x \in X$, $P(x)$ contains a minimal point [Au3,E], more precisely, every minimal subset of $\overline{Orb(x)}$ meets $P(x)$. It follows immediately that if $P(x)$ is a singleton then x is a minimal point and in the case x is called a *distal point*. Veech showed that a transitive system with a distal point has a very simple structure [V]. On the other hand, when the proximal cell is "big" the system will be complex in some sense. In [KR] the authors showed that in a weakly mixing system the set $\{x \in X : P(x) \text{ is residual in } X\}$ is residual in X. Hence a weakly mixing system is "almost proximal". Moreover, Furstenberg [F2] showed in a minimal weakly mixing system $P(x)$ is residual for all $x \in X$. Recently Akin and Kolyada gave an elegant proof and showed that this is true for any weakly mixing system [AK]. When one reads these papers, some nature questions come to mind. For example, for a dynamical system when will the converse of Akin-Kolyada's result hold? Can we say more concerning the structure of the proximal cell?

We discuss those questions in a more general setting: $\mathcal{F}$-mixing systems, where $\mathcal{F}$ is a collection of subsets of $\mathbb{Z}_+$ which is hereditary upward. We find that the notion of proximal cells along sequences is very useful in studying the questions. In Section 3 it is shown that if (X, T) is $\mathcal{F}$-mixing then for each $S \in k\mathcal{F}$ and $x \in X$, $P_S(x)$ is a dense G_δ set of X, and the reciprocal is true when (X, T) is minimal and $\mathcal{F}$ is a filter. Also some equivalence conditions for minimal weak mixing are listed. Lots of conditions when the converse holds are given in Section 4. In Section 5 the structure of the proximal cells of an $\mathcal{F}$-mixing system is discussed. Finally, in the Appendix a new and simpler proof of Xiong-Yang's theorem is given.

§2 Preliminary

In this section we introduce some basic notions and facts in TDS. Firstly we recall some notations related to a family. For the set of nonnegative integers $\mathbb{Z}_+$, denote by $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upward. That is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. For a family $\mathcal{F}$, the *dual family* is

$$k\mathcal{F} = \{F \in \mathcal{P} | F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}. \tag{2.1}$$

Sometimes the dual family $k\mathcal{F}$ is also denoted by $\mathcal{F}^*$. $k\mathcal{F}$ is a family, proper if $\mathcal{F}$ is.

Clearly, $k(k\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $k\mathcal{F}_2 \subset k\mathcal{F}_1$. Let $\mathcal{B}$ the family of all infinite subsets of $\mathbb{Z}_+$. And it is easy to see its dual $k\mathcal{B}$ is the family of all cofinite subsets.

A family $\mathcal{F}$ is *full* if $\mathcal{F} \cdot k\mathcal{F} \subset \mathcal{B}$, where $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2 : F_i \in \mathcal{F}_i, i = 1, 2\}$. If a proper family $\mathcal{F}$ satisfies $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$, then it is called a *filter*. If $\mathcal{F}$ is full then $k\mathcal{B} \subset \mathcal{F} \subset \mathcal{B}$. If $\mathcal{F}$ is a filter, then $k\mathcal{B} \subset \mathcal{F}$ implies $\mathcal{F}$ is full (see [Ak1]).

A subset F of $\mathbb{Z}_+$ is *thick* if it contains arbitrarily long runs of positive integers. Each element of the dual family of thick family is said to be *syndetic* or *relatively dense*. F is syndetic if and only if there is N such that $\{i, i+1, \cdots, i+N\} \cap F \neq \emptyset$ for every $i \in \mathbb{Z}_+$. A set F is called *thickly syndetic* if for every N the positions where length N runs begin form a syndetic set. And a set F is called *piecewise syndetic* if and only if it is the intersection of a thick set and a syndetic set. Among families above thickly syndetic family is a filter.

Let A be a subset of $\mathbb{Z}_+$. The *upper Banach density of* A is

$$d^*(A) = \limsup_{|I| \to \infty} \frac{|A \cap I|}{|I|}, \tag{2.2}$$

where I ranges over intervals of $\mathbb{Z}_+$ and $|\cdot|$ denote the cardinality of the set. The *upper density* of a subset A of $\mathbb{Z}_+$ is

$$\bar{d}(A) = \limsup_{N \to \infty} \frac{|A \cap \{0, 1, \cdots, N-1\}|}{N-1}. \tag{2.3}$$

The *lower Banach density* $d_*(A)$ and the *lower density* $\underline{d}(A)$ are similarly defined. If $\bar{d}(A) = \underline{d}(A)$, then we say A has density $d(A)$.

Let (X, T) be a TDS, $x \in X$ and $U, V \subset X$. We define the *return times set*

$$N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}(V) \neq \emptyset\}, \text{ and} \tag{2.4}$$

$$N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\} \tag{2.5}$$

Let family $\mathcal{F} \subset \mathcal{B}$. Recall that a TDS (X, T) is $\mathcal{F}$*-transitive* if for each pair of opene subsets U and V of X, $N(U, V) \in \mathcal{F}$. (X, T) is $\mathcal{F}$*-mixing* if $(X \times X, T \times T)$ is $\mathcal{F}$-transitive. $x \in X$ is called an $\mathcal{F}$*-recurrent point* if $N(x, U) \in \mathcal{F}$ for every neighborhood U of x. When we take $\mathcal{F} = \mathcal{B}$, $\mathcal{B}$-transitivity (respectively $\mathcal{B}$-mixing, $\mathcal{B}$-recurrence) is the usual transitivity (respectively weak mixing, recurrence).

A subset of $\mathbb{N}$ is an IP-set if it is equal to some $FS(\{p_i\}_{i=1}^{\infty}) = \{p_{i_1} + p_{i_2} + \cdots + p_{i_k} : k \in \mathbb{N}, 1 \leq i_1 < i_2 < \cdots < i_k\}$, where $p_i \in \mathbb{N}$. It is well known that (see, for example, [F2])

Lemma 2.1. *Let (X, T) be a TDS. Then*

1. *$x \in X$ is a minimal point iff for any neighborhood U of x, $N(x, U)$ is a syndetic set.*
2. *$x \in X$ is a recurrent point iff for any neighborhood U of x, $N(x, U)$ contains an IP-set.*

3. (X,T) *is weakly mixing iff* $N(U,V)$ *is thick for any opene sets* U,V *of* X*.*
4. (X,T) *is strongly mixing iff* $N(U,V)$ *is cofinite for any opene sets* U,V *of* X*.*

Finally, a TDS (X,T) is
- an *E-system* if it is transitive and there is an invariant Borel probability measure μ with full support, i.e., $supp(\mu)=X$;
- an *M-system* if it is transitive and the set of minimal points is dense; and
- a *P-system* if it is transitive and the set of periodic points is dense.

§3 $\mathcal{F}$-mixing systems and its proximal cells

Let (X,T) be a TDS with a metric d and $S\in\mathcal{B}$, the set of all infinite sequences of $\mathbb{Z}_+$. Recall that (x,y) is *S-proximal* if $\liminf_{S\ni n\to+\infty} d(T^n(x),T^n(y))=0$. Denote the S-proximal relation by P_S, i.e, $P_S=\{(x,y)\in X\times X:(x,y) \text{ is an } S-\text{proximal pair}\}$. Sometimes to indicate the space and the map we also use the notation $P_S(X,T)$. If $S=\mathbb{Z}_+$, $P_S(X,T)$ is just the ordinary proximal relation, by which we write $P(X,T)$ or P. Note that for a relation $R\subset X\times X$ and $x\in X$, $R(x)=\{y\in X:(x,y)\in R\}$. $P(x)$(resp. $P_S(x)$) is called the *proximal cell (resp. S-proximal cell)* at x. First we have the following easy observation.

Lemma 3.1. *Let* (X,T) *be a TDS and* $S=\{s_1<s_2<s_3<\cdots\}\in\mathcal{B}$*. Then*

$$P_S=\bigcap_{k=1}^{+\infty}\Big(\bigcup_{n=1}^{\infty}(T\times T)^{-s_n}\Delta_k\Big),$$

where $\Delta_k=\{(x,y):d(x,y)<\frac{1}{k}\}$*.*

Note that P_S and $P_S(x)$ are G_δ subsets, since Δ_k is an open set.

It is known that a family $\mathcal{F}$ is a filter iff $k\mathcal{F}$ has *Ramsey property*, i.e. if $\cup_{i=1}^n A_i\in k\mathcal{F}$ then one of A_i is still in $k\mathcal{F}$. Furstenberg [F1] showed that for a weakly mixing system (X,T) the smallest family containing $\{N(U,V):U,V \text{ are opene sets of } X\}$ is a filter. Hence for a full family $\mathcal{F}$, it is easy to see that (X,T) is $\mathcal{F}$-mixing iff it is weakly mixing and $\mathcal{F}$-transitive. Now we generalize the result of [AK] from a weakly mixing system to an $\mathcal{F}$-mixing system. Note that a detailed description of the proximal cells for $\mathcal{F}$-mixing systems is presented in Section 5.

Theorem 3.2. *Let* (X,T) *be a TDS and* $\mathcal{F}$ *be a full family. If* (X,T) *is* $\mathcal{F}$*-mixing then for each* $S\in k\mathcal{F}$ *and* $x\in X$*,* $P_S(x)$ *is a dense* G_δ *set of* X*.*

Proof. The proof is close to that of Theorem 3.8 in [AK]. If (X,T) is trivial, it is obvious. Now we suppose that (X,T) is non-trivial. Let $\mathcal{F}_1$ be the filter generated by $\{N(U,V):U,V \text{ are opene sets of } X\}$. Then $\mathcal{F}_1\subset\mathcal{F}$ and hence $k\mathcal{F}_1\supset k\mathcal{F}$. Since (X,T) is non-trivial $\mathcal{F}$-mixing, $k\mathcal{B}\subset\mathcal{F}_1$. Hence $(k\mathcal{B}\cdot\mathcal{F}_1)\cdot k\mathcal{F}_1=\mathcal{F}_1\cdot k\mathcal{F}_1$, therefore

$\mathcal{F}_1$ is full family. For $S_1 \in k\mathcal{F}_1$, given $x \in X$ and an opene set U of X we will find some $y \in U$ such that y is S_1-proximal to x.

For $k = 1, 2, \dots$ let $\mathcal{G}_k = \{G_k^1, \dots, G_k^{n_k}\}$ be a finite open cover of X with the diameter less than $1/k$. As

$$S_1 = S_1 \cap N(x, X) = \bigcup_{j=1}^{n_k} (S_1 \cap N(x, G_k^j))$$

there is some $i \in \{1, 2, \cdots, n_k\}$ such that $S_1 \cap N(x, G_k^i)$ is in $k\mathcal{F}_1$, since $\mathcal{F}_1$ is a filter and hence $k\mathcal{F}_1$ has Ramsey property.

Now let $U_0 = U$ and define inductively opene sets $U_1, U_2, \cdots$ and positive integers $n_j \in S_1$ as follows. Since $N(U_{k-1}, G_k^i) \in \mathcal{F}_1$, $S_1 \cap N(x, G_k^i) \cap N(U_{k-1}, G_k^i) \in \mathcal{B}$ and we can choose an opene set $U_k \subset U_{k-1}$ and an integer $k < n_k \in S_1$ such that $T^{n_k}(x) \in G_k^i$ and $T^{n_k}(\overline{U_k}) \subset G_k^i$. If $y \in \bigcap_k \overline{U_k}$, then $T^{n_k}(x), T^{n_k}(y) \in G_k^i$ and so $d(T^{n_k}(x), T^{n_k}(y)) < 1/k$. Thus $y \in U$ is S_1 proximal to x. $\square$

If (X, T) is strongly mixing Theorem 3.2 states that for each $S \in \mathcal{B}$ and each $x \in X$, $P_S(x)$ is a dense G_δ-set! We now show that the converse of Theorem 3.2 holds when $\mathcal{F}$ is a filter and X is minimal. Especially, for a minimal TDS (X, T), $P_S(x)$ is dense for each $S \in \mathcal{B}$ and $x \in X$ iff (X, T) is strongly mixing.

To show what we just claimed, we need to introduce some notations. Let (X, T) be a TDS, $X^n = X \times X \times \cdots \times X$ (n times) be product system, $S = \{s_1, s_2, \dots\} \in \mathcal{B}$ and $n \geq 2$. Set

$$RP_S^n(X,T) = RP_S^n =: \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} (T^{(n)})^{-s_n} \Delta_{\frac{1}{k}}^{(n)}} \tag{3.1}$$

where $T^{(n)} = T \times T \times \cdots \times T$ (n times) and $\Delta_\epsilon^{(n)} = \{(x_1, x_2, \cdots, x_n) \in X^n : \sup_{1 \leq k,l \leq n} d(x_k, x_l) \leq \epsilon\}$. It is easy to check that $x \in RP_S^n$ iff there exist $x_i' \in X^n$ with $x_i' \to x$ and $\{n_i\} \subset S$ such that $T^{n_i} x_i' \to \Delta^{(n)}$, where $\Delta^{(n)} = \{(x, x, \cdots, x) \in X^n : x \in X\}$. For a family $\mathcal{F}$, let $RP_{\mathcal{F}}^n(X,T) = \bigcap_{S \in k\mathcal{F}} RP_S^n(X,T)$.

The following proposition first appears in [HY2] and we include a proof for completeness.

Proposition 3.3. *Let (X, T) be a minimal TDS and $\mathcal{F}$ be a full family. Then (X, T) is $\mathcal{F}$-mixing iff $RP_{\mathcal{F}}^n(X,T) = X^n$ for each $n \geq 2$.*

Proof. First we assume $RP_S^n(X,T) = X^n$ for each $n \geq 2$ and $S \in k\mathcal{F}$. As $RP_S^2(X,T) = X^2$ for $S \in k\mathcal{F} \subset \mathcal{B}$, one has that the regionally proximal relation $RP_{\mathbb{Z}_+}^2(X,T) = X^2$. Thus (X, T) is weakly mixing from the minimality of (X, T) (see [Au3]). Hence it remains to show (X, T) is $\mathcal{F}$- transitive, i.e. for any two opene subset U and V of X, $N(U, V) \in \mathcal{F}$.

As (X,T) is minimal, there exists $N \in \mathbb{N}$ such that $\bigcup_{n=0}^{N-1} T^{-n}V = X$. Let $\delta > 0$ be the Lebesgue number of the open cover $\{V, T^{-1}V, \cdots T^{-(N-1)}V\}$. Define

$$A = \{n \in \mathbb{Z}_+ : \exists x_i \in T^{-i}U, i = 0,1,\cdots,N-1 \text{ such that } \sup_{0 \le k,l \le N-1} d(T^n x_k, T^n x_l) \le \delta\}.$$

Since $RP_S^N(X,T) \cap (U \times T^{-1}U \times \cdots \times T^{N-1}U) \neq \emptyset$ for each $S \in k\mathcal{F}$, one has $A \cap S \neq \emptyset$ for each $S \in k\mathcal{F}$. Hence $A \in \mathcal{F}$. For any $n \in A$, there exists $x_i \in T^{-i}U, i = 0,1,\cdots,N-1$ such that $\mathrm{diam}\{T^n x_0, T^n x_1, \cdots, T^n x_{N-1}\} \le \delta$. Therefore, $\{T^n x_0, T^n x_1, \cdots, T^n x_{N-1}\} \subset T^{-k}V$ for some $0 \le k \le N-1$. In particular, $x_k \in T^{-(n+k)}V$. Hence $T^{-k}U \cap T^{-(n+k)}V \neq \emptyset$, that is, $T^{-n}V \cap U \neq \emptyset$. This shows that $A \subset N(U,V)$ and implies that $N(U,V) \in \mathcal{F}$. As U, V are arbitrary, (X,T) is $\mathcal{F}$-transitive.

Now assume (X,T) is $\mathcal{F}$-mixing. Let $n \in \mathbb{N}$ and $S \in k\mathcal{F}$. And let $x_i \in X$ and U_i be a neighborhood of x_i, $1 \le i \le n$. For any $\epsilon > 0$ take an opene U with $diamU < \epsilon$. As (X,T) is $\mathcal{F}$- mixing, $\bigcap_{i=1}^{n} N(U_i, U) \in \mathcal{F}$, and thus we can take $m \in \bigcap_{i=1}^{n} N(U_i, U) \bigcap S$. Hence there are $x_i' \in U_i, 1 \le i \le n$ such that $\{T^m x_i' : 1 \le i \le n\} \subset U$, i.e. $\sup_{1 \le k,l \le n} d(T^m x_k', T^m x_l') \le \epsilon$. Thus we have $(x_1, x_2, \cdots, x_n) \in RP_S^n(X,T)$. That is, $RP_S^n(X,T) = X^n$. □

Now we are ready to show

Theorem 3.4. *Let (X,T) be minimal and $\mathcal{F}$ be a filter. Then (X,T) is $\mathcal{F}$-mixing iff for each $S \in k\mathcal{F}$ and $x \in X$, $P_S(x)$ is a dense G_δ-set of X.*

Proof. It remains to show that if for each $S \in k\mathcal{F}$ and $x \in X$, $P_S(x)$ is a dense G_δ-set of X, then $RP_S^n(X,T) = X^n$ for each $n \ge 2$. Now let $U_1, \ldots, U_n$ be opene sets of X. Let $x_1 \in U_1$. Given $\epsilon > 0$ let

$$A_i = \{k \in \mathbb{Z}_+ : \text{there is } x_i \in U_i \text{ with } d(T^k(x_1), T^k(x_i)) < \epsilon/2\}$$

for $i = 2, \ldots, n$. As $P_S(x_1)$ is dense for each $S \in k\mathcal{F}$, we get that $A_i \in \mathcal{F}, 2 \le i \le n$.

For a given $S \in k\mathcal{F}$, $A_2 \cap \ldots \cap A_n \cap S \neq \emptyset$. Thus there are $x_i \in U_i$ and $n \in S$ such that

$$d(T^n(x_1), T^n(x_i)) < \epsilon/2$$

for $i = 2, \ldots, n$. This implies that $\sup_{1 \le i < j \le n} d(T^n(x_i), T^n(x_j)) < \epsilon$, and hence $RP_S^n(X,T) = X^n$ for each $S \in k\mathcal{F}$ and $n \ge 2$. By Proposition 3.3 this implies that (X,T) is $\mathcal{F}$-mixing. □

Corollary 3.5. *Let (X,T) be a minimal TDS and for each $S \in \mathcal{B}$ and $x \in X$, $P_S(x)$ is dense in X. Then (X,T) is strongly mixing.*

Proof. Let $k\mathcal{F} = \mathcal{B}$. Then the corollary follows from Theorem 3.4. □

To end the section we now state several equivalence conditions for a TDS to be minimal weakly mixing. To this aim we first need some more notations. Let (X,T) be a TDS and its enveloping semigroup $E = E(X,T)$ be the closure of $\{T^n : n \in \mathbb{Z}_+\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). An $u \in E(X,T)$ with $u^2 = u$ is called an *idempotent*. The set of all idempotents of $E(X,T)$ is denoted by $Id(E(X,T))$. Ellis pointed out that for any TDS the idempotents in $E(X,T)$ always exist. A non-empty subset $I \subset E$ is a *left ideal* if it is closed and $EI \subset I$. A *minimal left ideal* is a left ideal which does not contain any proper left ideal of E. Obviously every left ideal is semigroup and every left ideal contains some minimal ideal. An idempotent is *minimal* if it is contained in some minimal left ideal (for details concerning the enveloping semigroup see, for example, [Au3,E]). It is well known that for a minimal system (X,T) x is proximal to y iff there is some minimal idempotent u such that $y = ux$. And hence for a minimal system $P(x) = Id(E(X,T))x = \{ux : \ u \text{ is a minimal idempotent of } E(X,T)\}$.

Now we give some equivalence conditions for a system to be minimal weak mixing which are closely related to our paper. For other equivalence conditions, see for example [Au3], [HY2].

Theorem 3.6. *Let (X,T) be a minimal system. Then the following statements are equivalent:*

(1) (X,T) is weakly mixing.
(2) (X,T) is thickly syndetic transitive.
(3) For any $x \in X$, $P(x)$ is dense in X.
(4) For any $x \in X$ and any piecewise syndetic set S, $P_S(x)$ is dense in X.
(5) For any $x \in X$, $Id(E(X,T))x$ is dense in X.
(6) For any $x \in X$, $\{ux : u$ is a minimal idempotent of $E(X,T)\}$ is dense in X.
(7) For any $x \in X$ and any opene set U of X, $N(x,U)$ contains an IP set.

Proof. (2) ⇒ (4): It follows from Theorem 3.4.

(4) ⇒ (3): It is obvious.

(3) ⇒ (1): It follows from Theorem 3.4.

(1) ⇒ (2): As (X,T) is minimal it is syndetic transitive, i.e. $N(U,V)$ is syndetic for any opene sets U,V of X. Since (X,T) is weakly mixing, the smallest family containing $\{N(U,V) : U,V$ are opene sets of $X\}$ is a filter [F1]. This implies that for each $n \in \mathbb{N}$ and opene sets U,V of X, $\bigcap_{i=0}^{n-1} N(U,T^{-i}V)$ contains $N(U',V')$ for some opene U',V', it is syndetic, thus $N(U,V)$ is thickly syndetic.

(3) ⇒ (6): Assume that for any $x \in X$, $P(x)$ is dense in X. Let $x \in X$ and U be any opene set of X. We show there is some minimal idempotent u such that $ux \in U$.

Since $P(x)$ is dense in X, there is $y \in P(x) \cap U$. As (X,T) is minimal, by the fact mentioned before there is some minimal idempotent u such that $y = ux$. Thus for any opene set U of X there is some minimal idempotent u such that $ux \in U$, i.e. (6) holds.

(6) $\Rightarrow$ (5): It is obvious.

(5) $\Rightarrow$ (7): Let $x \in X$ and U be any opene set of X. As (5) holds there is some idempotent u such that $y = ux \in U$. Hence for any neighborhood V of y, $N(x,V) \cap N(y,V) \in \mathcal{B}$. Let $U_0 = U$. Take natural number $p_1 \in N(x,U_0) \cap N(y,U_0)$ and $U_1 = U_0 \cap T^{-p_1}U_0$. Then $y \in U_1$. By induction, we can construct $p_i, U_i, i \in \mathbb{N}$ such that natural number $p_i \in N(x,U_{i-1}) \cap N(y,U_{i-1})$ and $y \in U_i = U_{i-1} \cap T^{-p_{i-1}}U_{i-1}$. By the above construction, it is easy to that $N(x,U) \supset FS(\{p_i\}_{i=1}^{+\infty})$ and hence $N(x,U)$ contains an IP-set.

(7) $\Rightarrow$ (1): Let U_1, U_2, V_1, V_2 be opene subsets of X. Then by the definition

$$N(U_1 \times V_1, U_2 \times V_2) = \{n \in \mathbb{Z}_+ : (T \times T)^{-n}(U_2 \times V_2) \cap (U_1 \times V_1) \neq \emptyset\}.$$

As (X,T) is minimal , there is $k \in \mathbb{Z}_+$ such that $V = T^{-k}V_2 \cap V_1$ is a opene subset of X. Thus,

$$\begin{aligned} & N(U_1 \times V_1, U_2 \times V_2) \\ = & \{n \in \mathbb{Z}_+ : (T \times T)^{-n}(U_2 \times V_2) \cap (U_1 \times V_1) \neq \emptyset\} \\ \supset & k + \{m \in \mathbb{Z}_+ : (T^{-(m+k)}U_2 \cap U_1) \times (T^{-(m+k)}V_2 \cap V_1) \neq \emptyset\} \\ \supset & k + \{m \in \mathbb{Z}_+ : (T^{-m}T^{-k}U_2 \cap U_1) \times (T^{-m}V \cap V) \neq \emptyset\} \\ = & k + N(U_1, T^{-k}U_2) \cap N(V,V). \end{aligned}$$

Let $x \in U_1$. Then $N(U_1, T^{-k}U_2) \supset N(x, T^{-k}U_2)$ contains an IP-set, generated by $\{p_1, p_2, \dots\}$. Take any invariant measure μ of (X,T). As (X,T) is minimal, one has $\mu(V) > 0$.

Claim: Let A be a Borel set with $\mu(A) > 0$. Then for any IP-set S, there exists $n \in S$ such that $\mu(A \cap T^{-n}A) > 0$.

Proof of claim. Let S be an IP-set generated by $p_1, p_2, p_3, \cdots$. Since μ is a probability measure and $\mu(T^{-(p_1+p_2+\cdots+p_j)}A) = \mu(A) > 0$ for each $j \in \mathbb{N}$, there exist $j_1 < j_2$ such that $\mu(T^{-(p_1+p_2+\cdots+p_{j_1})}A \cap T^{-(p_1+p_2+\cdots+p_{j_2})}A) > 0$. Thus $\mu(A \cap T^{-n}A) > 0$ for $n = p_{j_1+1} + \cdots + p_{j_2}$. This ends the proof of claim.

By the above claim, there is $n \in N(U_1, T^{-k}U_2)$ with $\mu(V \cap T^{-n}V) > 0$, i.e, $n \in N(V,V)$. Hence $N(U_1, T^{-k}U_2) \cap N(V,V) \neq \emptyset$. Moreover, $N(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$. As U_1, U_2 and V_1, V_2 are arbitrary, $(X \times X, T \times T)$ is transitive. This shows that (X,T) is weakly mixing. $\square$

Remark: Among the mixing properties, weak mixing is the weakest one and strong mixing is the strongest one. Now we have that in the minimal case weak mixing is equivalent to thickly syndetic transitivity and strong mixing is equivalent to cofinite transitivity. As the set of all thickly syndetic set and the set of all cofinite set are both filters, this in some sense explains why we require $\mathcal{F}$ to be a filter in Theorem 3.4.

§4 OTHER CONDITIONS FOR WHICH THE CONVERSE OF THEOREM 3.2 HOLDS

In the previous section we have shown that for a full family $\mathcal{F}$ if a TDS (X, T) is $\mathcal{F}$-mixing then for each $x \in X$ and each $S \in k\mathcal{F}$, $P_S(x)$ is dense, and the converse holds when (X, T) is minimal and $\mathcal{F}$ is a filter. In this section we will give more conditions such that the converse of Theorem 3.2 holds. First we discuss the situation when (X, T) is an E-system.

To do this, first let us recall some notion. Let $\{p_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $FS(\{p_i\}_{i=1}^{\infty}) := \{p_{i_1} + p_{i_2} + \cdots + p_{i_k} : k \in \mathbb{N}, 1 \le i_1 < i_2 < \cdots < i_k\}$. A set $A \subset \mathbb{N}$ is called an *IP-set* if it equals to some $FS(\{p_i\}_{i=1}^{\infty})$. Denote the family generated by all IP-sets by $\mathcal{F}_{IP}$ or IP. By Hindman' Theorem [H] $\mathcal{F}_{IP}$ has Ramsey property and hence its dual family $\mathcal{F}_{IP}^*$ is a filter (every element in $\mathcal{F}_{IP}^*$ is called an *IP^*-set*).

A TDS (X, T) is said to be *mild mixing* if it is weakly disjoint from any transitive system, i.e. for any transitive system (Y, S), $(X \times Y, T \times S)$ is transitive. For a subset S of $\mathbb{Z}_+$, let $S - S = \{s - t : s, t \in S \text{ and } s \ge t\}$. If $\mathcal{F}$ is a family let $\mathcal{F} - \mathcal{F} = \{S - S : S \in \mathcal{F}\}$. It is showed in [HY2] that (X, T) is mild mixing if and only if it is $k(\mathcal{F}_{IP} - \mathcal{F}_{IP})$-transitive. Consequently, IP*-transitivity implies mild mixing.

The following lemma which is a special case of a Furstenberg and Katznelson's theorem [FK] will be used.

Lemma 4.1. *Let $(X, T, \mathcal{B}, \mu)$ be an invertible m.p.s.. and A be a Borel set with $\mu(A) > 0$. Then for any IP-set S, there exists $n \in S$ such that $\mu(A \cap T^{-n}A \cap T^nA) > 0$.*

Now we are ready to show.

Theorem 4.2. *Let (X, T) be a TDS, and assume that (X, T) has an invariant measure μ with Supp$(\mu) = X$. Then*
1. (X, T) is strongly mixing iff for each $S \in \mathcal{B}$ and each $x \in X$, $P_S(x)$ is dense.
2. (X, T) is IP*-mixing iff for each IP-set S and each $x \in X$, $P_S(x)$ is dense.
3. (X, T) is mild-mixing iff for each IP-set S and each $x \in X$, $P_{S-S}(x)$ is dense.

Proof. (1) It remains to show that if for each $S \in \mathcal{B}$ and each $x \in X$, $P_S(x)$ is dense, then (X, T) is strongly mixing. Let U, V be opene sets of X. Take an opene set $V_1 \subset V$ with $\overline{V_1} \subset V$. Let $S = \{s_1, s_2, \dots\} \in \mathcal{B}$ and $A_k = \cup_{i=k}^{\infty} T^{-s_i}(V_1)$. Then A_k is a decreasing sequence of open sets on k. As there is an invariant measure μ with full support, $\mu(A_k) \ge \mu(V_1)$, $k = 1, 2, \dots$. This implies that $\mu(\bigcap_{k=1}^{\infty} A_k) \ge \mu(V_1) > 0$

and hence $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$. Take $x \in \bigcap_{k=1}^{\infty} A_k$. Thus there is $S_1 \subset S$ and $S_1 \in \mathcal{B}$ such that $T^n(x) \in V_1$ for $n \in S_1$.

As $P_{S_1}(x)$ is dense, there are $y \in U$, $S_2 \subset S_1$ and $S_2 \in \mathcal{B}$ such that $\lim_{S_2 \ni n \to \infty} d(T^n(x), T^n(y)) = 0$. This implies that there is $n \in S$ such that $T^n(y) \in V$. Consequently, $N(U, V) \cap S \neq \emptyset$. Since $S \in \mathcal{B}$ is arbitrary, this implies that $N(U, V)$ is cofinite, i.e. (X, T) is strongly mixing.

(2) It remains to show that if for each IP-set S and each $x \in X$, $P_S(x)$ is dense, then (X, T) is IP*-transitive. First, we have

Let U, V be opene subsets of X. As $\mathrm{Supp}(\mu) = X$, $\mu(V) > 0$ and there exists a closed set $A_1 \subset V$ with $\mu(A_1) > 0$. For any IP-set S, generated by $p_1, p_2, p_3, \cdots$, by claim there exists $q_1 = p_{j_1^1} + \cdots + p_{j_{k_1}^1}$ with $j_1^1 < j_2^1 < \cdots < j_{k_1}^1$ such that $\mu(A_1 \cap T^{-q_1} A_1) > 0$.

Set $A_2 = A_1 \cap T^{-q_1} A_1$ and let S_1 be the IP-set generated by $p_{j_{k_1}^1+1}, p_{j_{k_1}^1+2}, \cdots$. Then $A_2 \subset A_1$ is a closed subset of X with $\mu(A_2) > 0$. By a similar argument, there exists $q_2 = p_{j_1^2} + \cdots p_{j_{k_2}^2}$ with $j_{k_1}^1 + 1 \leq j_1^2 < j_2^2 < \cdots < j_{k_2}^2$ such that $\mu(A_2 \cap T^{-q_2} A_2) > 0$.

By induction, we can construct $A_{l+1} = A_k \cap T^{-q_l} A_k$ and $q_{l+1} = p_{j_1^{l+1}} + \cdots + p_{j_{k_{l+1}}^{l+1}}$ with $j_{k_l}^l + 1 \leq j_1^{l+1} < j_2^{l+1} < \cdots < j_{k_{l+1}}^{l+1}$ such that $A_{l+1} \subset A_l$ is a closed subset with $\mu(A_{l+1} \cap T^{-q_{l+1}} A_{l+1}) > 0$.

Now as $\mu(A_k) > 0$, A_k is a non-empty closed set. Noting that $A_1 \supset A_2 \supset A_3 \supset \cdots$, one has $\bigcap_{k=1}^{+\infty} A_k \neq \emptyset$. Let S' be the IP-set generated by $q_1, q_2, q_3, \cdots$. Then $S' \subset S$.

Choose $x \in \bigcap_{k=1}^{+\infty} A_k$. It is easy to see that $x \in \bigcap_{n \in S'} T^{-n} A_1$. Hence $N(x, A_1) \supset S'$. As $P_{S'}(x)$ is dense, there are $y \in U$, $S'' \subset S'$ such that $\lim_{S'' \ni n \to \infty} d(T^n(x), T^n(y)) = 0$. This implies that there is $n \in S$ such that $T^n(y) \in V$. Consequently, $N(U, V) \cap S \neq \emptyset$. This implies that $N(U, V)$ is an IP*-set, i.e. (X, T) is IP*-transitive. Moreover, as the family consisting of IP*-sets is a filter, (X, T) is IP*-mixing.

(3) Without loss of generality, we assume that T is a homeomorphism. It remains to show that if for each IP-set S and each $x \in X$, $P_{S-S}(x)$ is dense, then (X, T) is mild mixing.

Let U, V be opene subsets of X. As $\mathrm{Supp}(\mu) = X$, $\mu(V) > 0$, there exists a closed set $A_1 \subset V$ with $\mu(A_1) > 0$. For any IP-set S, generated by $p_1, p_2, p_3, \cdots$, by Lemma 4.1 there exists $q_1 = p_{j_1^1} + \cdots + p_{j_{k_1}^1}$ with $j_1^1 < j_2^1 < \cdots < j_{k_1}^1$ such that $\mu(A_1 \cap T^{-q_1} A_1 \cap T^{q_1} A_1) > 0$.

Set $A_2 = A_1 \cap T^{-q_1} A_1 \cap T^{q_1} A_1$ and let S_1 be the IP-set generated by $p_{j_{k_1}^1+1}, p_{j_{k_1}^1+2}, \cdots$. Then $A_2 \subset A_1$ is closed subset of X with $\mu(A_2) > 0$. Similarly, there exists $q_2 = p_{j_1^2} + \cdots p_{j_{k_2}^2}$ with $j_{k_1}^1 + 1 \leq j_1^2 < j_2^2 < \cdots < j_{k_2}^2$ such that $\mu(A_2 \cap T^{-q_2} A_2 \cap T^{q_2} A_2) > 0$.

By induction, we can construct $A_{l+1} = A_l \cap T^{-q_l} A_l \cap T^{q_l} A_l$ and $q_{l+1} = p_{j_1^{l+1}} + \cdots + p_{j^{l+1}_{k_{l+1}}}$ with $j^l_{k_l} + 1 \leq j^{l+1}_1 < j^{l+1}_2 < \cdots < j^{l+1}_{k_{l+1}}$ such that $A_{l+1} \subset A_l$ is a closed subset with $\mu(A_{l+1} \cap T^{-q_{l+1}} A_{l+1} \cap T^{q_{l+1}} A_{l+1}) > 0$.

Now as $\mu(A_k) > 0$, A_k is a non-empty closed set. Noting that $A_1 \supset A_2 \supset A_3 \supset \cdots$, one has $\bigcap_{k=1}^{+\infty} A_k \neq \emptyset$. Let S' be the IP-set generated by $q_1, q_2, q_3, \cdots$. Then $S' \subset S$.

Take $x \in \bigcap_{k=1}^{+\infty} A_k$. It is easy to see that $x \in \bigcap_{n \in S'-S'} T^{-n} A_1$. Hence $N(x, A_1) \supset S' - S'$. As $P_{S'-S'}(x)$ is dense, there are $y \in U$, $S'' \subset S'$ such that $\lim_{S'' \ni n \to \infty} d(T^n(x), T^n(y)) = 0$. This implies that there is $n \in S$ such that $T^n(y) \in V$. Consequently, $N(U, V) \cap (S - S) \neq \emptyset$. This implies that $N(U, V)$ is an $k(\mathcal{F}_{IP} - \mathcal{F}_{IP})$-set, i.e. (X, T) is mild mixing. □

Now we discuss the question under which conditions a TDS is weakly mixing.

A TDS is said to be *totally transitive* if (X, T^n) is transitive for all $n \in \mathbb{N}$. It is known that if (X, T) is minimal then (X, T) is weakly mixing iff for each $x \in X$, $P(x)$ is dense (see previous section). Now we have

Theorem 4.3. *Let (X, T) be a TDS.*

1 *If (X, T) has dense minimal points and $P(x)$ is dense for each $x \in X$ then (X, T) is weakly mixing.*
2 *If (X, T) is totally transitive and for each open U there is $p \in \mathbb{N}$ such that $N(U, U) \supset p\mathbb{N}$, then (X, T) is weakly mixing. Consequently, if (X, T) is totally transitive and the set of periodic points is dense then (X, T) is weakly mixing.*

Proof. (1). If (x, y) is proximal, then for any $\epsilon_1 > 0$, $\{n \in \mathbb{Z}_+ : d(T^n(x), T^n(y)) < \epsilon_1\}$ is a thick set. It follows that if (x, y) is proximal, then (x, y) is S-proximal for any syndetic set of $\mathbb{Z}_+$. Let U, V be open subsets of X. As (X, T) has dense minimal points, there is a minimal point $x \in V$ and $B_\epsilon(x) \subset V$ for some $\epsilon > 0$. Set $V_1 = B_{\epsilon/2}(x)$. It is clear that $S = N(x, V_1)$ is syndetic. There is $y \in U$ such that (x, y) is S-proximal. That is, there is $n \in S$ such that $d(T^n(x), T^n(y)) \leq \epsilon/2$. As $d(T^n(x), x) < \epsilon/2$, we conclude that $T^n(y) \in V$. As $x \in V$ and $T^n(x) \in V$, $n \in \mathbb{N}(V, V)$. Thus $n \in N(V, V) \cap N(U, V)$. This implies that $N(V, V) \cap N(U, V) \neq \emptyset$ for each opene subset U, V of X and hence (X, T) is transitive.

Now we show that (X, T) is weakly mixing. Let U_1, U_2, V_1, V_2 be opene subsets of X. It is similar to the proof of (7) $\Rightarrow$ (1) in Theorem 3.6, there exists $k \in \mathbb{Z}_+$ such that $V' = T^{-k} V_2 \cap V_1$ is a opene subset of X and

$$N(U_1 \times V_1, U_2 \times V_2) \supset k + N(U_1, T^{-k} U_2) \cap N(V', V').$$

Since (X, T) is transitive, there is $m \in \mathbb{Z}_+$ such that $T^{-(m+k)} U_2 \cap V' \neq \emptyset$. Let $V = T^{-(m+k)} U_2 \cap V'$ and $U = T^m U_1$. Then

$$\begin{aligned} N(U_1, T^{-k} U_2) \cap N(V', V') &= N(T^{-m} U_1, T^{-(m+k)} U_2) \cap N(V', V') \\ &\supset N(U, V) \cap N(V, V) \neq \emptyset. \end{aligned}$$

Therefore $N(U_1 \times V_1, U_2 \times V_2) \neq \emptyset$. Hence (X,T) is weakly mixing.

(2). Let U, V be open subsets of X. As (X,T) is totally transitive, for each $p \in \mathbb{N}$, $N(U,V) \cap p\mathbb{N} \neq \emptyset$. Thus $N(U,U) \cap N(U,V) \neq \emptyset$ and hence (X,T) is weakly mixing. □

The following example shows that the assumption of Theorem 4.3 is reasonable.

Example 4.4. *There exists an E-system (X,T) such that $P(x)$ is dense for each $x \in X$, but (X,T) is not weakly mixing. Moreover, a totally transitive M-system is not necessarily weakly mixing.*

Proof. Let (Y,S) be an E-system but not an M-system and be not weakly mixing. Collapsing the closure of minimal points of (Y,S) to a point, we get a factor system (X,T) of (Y,S). (X,T) is a non-trivial E-system, has only a unique minimal point which is a fixed point and is not weakly mixing. Since (X,T) has a unique minimal point which is a fixed point, so any pair of $X \times X$ is proximal.

Now let us see how to obtain such a system (Y,S). Choose a set A with positive upper Banach density but being not piecewise syndetic. Let $X = \overline{Orb(1_A, \sigma)} \subseteq \{0,1\}^{\mathbb{Z}_+}$, where 1_A is the indication function of A. Then as A has positive upper Banach density there is some ergodic measure ν with $\nu([1]) > 0$, where $[1] = \{w \in X : w(0) = 1\}$ (see Proposition 3.17 of [F2]). Let $Y = supp\nu$ and $S = \sigma$. Then (Y,S) is an $E-$system. Since A is not piecewise syndetic, (Y,S) has $(0,0,\cdots)$ as its only minimal set and hence it is not an M-system. If (Y,S) is not weakly mixing we are done. If it is, then we product it with a minimal irrational rotation of the circle. Then the resulting system is a non-weakly mixing system and it is still an $E-$system but not an $M-$ system.

Let S^1 be the unit circle in the complex plane and T be a rotation so that (S^1,T) is a minimal system. It is a totally transitive M-system is not weakly mixing. □

Example 4.4 tells us that for an E-system Theorem 4.3 is no longer true. Nevertheless, we can get similar results by strengthening the assumption on proximal cells. First, we express Theorem 4.3 in another form. To do this, we note that the following two facts.

Fact 1. (x,y) is proximal iff (x,y) is S-proximal for any syndetic set S.

Fact 2. A point $x \in X$ is syndetic recurrent iff it is a minimal point. Moreover, (X,T) has dense minimal points iff (X,T) has dense syndetic recurrent points.

By above two facts, Theorem 4.3 can be expressed as follows: if (X,T) has dense syndetic recurrent points and $P_S(x)$ is dense for each $x \in X$ and syndetic set S then (X,T) is weakly mixing. In general, we have

Theorem 4.5. *Let (X,T) be a TDS and $\mathcal{F}$ be a family. If the set of all $\mathcal{F}$-recurrent points is dense in X and $P_S(x)$ is dense for each $x \in X$ and $S \in \mathcal{F}$, then (X,T) is weakly mixing.*

Proof. Let U, V be opene subsets of X. As (X,T) has dense $\mathcal{F}$-recurrent points, there is a $\mathcal{F}$-recurrent point $x \in V$ and $B_\epsilon(x) \subset V$. Set $V_1 = B_{\epsilon/2}(x)$. It is clear that $S = N(x, V_1) \in \mathcal{F}$. There is $y \in U$ such that (x, y) is S-proximal. That is, there is $n \in S$ such that $d(T^n(x), T^n(y)) \leq \epsilon/2$. As $d(T^n(x), x) < \epsilon/2$, we conclude that $T^n(y) \in V$. As $x \in V$ and $T^n(x) \in V$, $n \in N(V, V)$. This implies that $N(V, V) \cap N(U, V) \neq \emptyset$ and hence (X, T) is weakly mixing. □

The following two corollaries are immediately.

Corollary 4.6. *Let (X, T) be a TDS. If (X, T) has an invariant measure μ with $Supp(\mu) = X$ and $P_S(x)$ is dense for each $x \in X$ and positive upper density set S, then (X, T) is weakly mixing.*

Proof. By Theorem 4.5, it remains to show that (X, T) has dense positive upper density-recurrent points. Let $\mu = \int_\Omega \mu_\omega dm(\omega)$ be the ergodic decomposition of μ. Given an opene subset U of X, as $\mu(U) > 0$, there exists an ergodic measure μ_ω with $\mu_\omega(U) > 0$. Take a generic point $y \in U$ for μ_ω. Then y is a positive upper density-recurrent point. As U is arbitrary, (X, T) has dense positive upper density-recurrent points. □

Corollary 4.7. *Let (X, T) be a TDS and $R(T)$ be the set of all recurrent points of X. If $\overline{R(T)} = X$ and $P_S(x)$ is dense for each IP-set $S \subset \mathbb{Z}_+$ and each $x \in X$, then (X, T) is weakly mixing.*

Proof. Note that if $\overline{R(T)} = X$, then (X, T) has dense IP-recurrent points by Lemma 2.1. Then corollary 4.7 follows from Theorem 4.5. □

Remark 4.8. *Without the assumption of $\overline{R(T)} = X$ Corollary 4.7 may be false. For example, let T be a translation of $\mathbb{Z}$. Then the induced map T on the one point compactification of $\mathbb{Z}$ satisfies the condition ($\star$): $P_S(x)$ is dense for each $S \in \mathcal{B}$ and each $x \in X$, but it is not weakly mixing, even not transitive.*

It is an open question if a system satisfying the assumption of condition $\star$ and transitivity is strongly mixing.

§5 The structure of the proximal cells

We discuss the structure of the proximal cells of $\mathcal{F}$-mixing systems in this section. In [HY1] the authors showed that if a non-periodic transitive system contains a periodic point, then there is an uncountable scrambled set. Recently Mai [M] gave a constructive proof of the fact. Inspired by his method, we now describe the structure of proximal cells of $\mathcal{F}$- mixing systems, which deeps our understanding of proximal cells.

First we introduce some notions appeared in our theorem. If X, Y are topological spaces, then we denote by $\mathcal{C}(X, Y)$ the set of all continuous maps from X to Y.

Definition 5.1. *Let (X, T) be a TDS and $S \in \mathcal{B}$. A subset C of X is called a* **Kronecker subset with respect to** *S if C is a Cantor set and for every $g \in$*

$\mathcal{C}(C,X)$ *and* $\epsilon>0$ *there exists a positive integer* $n\in S$ *such that* $d(g(x),T^n(x))<\epsilon$ *for all* $x\in C$, *i.e.* $\{T^n|_C : n\in S\}$ *is uniformly dense in* $\mathcal{C}(C,X)$.

A subset K *of* X *is a* **chaotic set with respect to** S *if for every* $g\in\mathcal{C}(K,X)$ *there is a subsequence* $\{q_i\}\subset S$ *such that* $\lim\limits_{i\to\infty}T^{q_i}(x)=g(x)$ *for every* $x\in K$, *i.e.* $\{T^n|_K : n\in S\}$ *is pointwise dense in* $\mathcal{C}(K,X)$.

By the above definition, it is easy to see that a Kronecker subset with respect to S must be a chaotic set with respect to S.

Theorem 5.2. *Let* (X,T) *be a non-trivial TDS and* $\mathcal{F}$ *be a full family. If it is* $\mathcal{F}$*-mixing, then for any* $x\in X$ *and any* $S\in k\mathcal{F}$ *there are Cantor sets* $C_1\subseteq C_2\subseteq\cdots$ *such that:*

(i) $K=\bigcup\limits_{n=1}^{\infty}C_n$ *is dense in* X.

(ii) There are $\{k_n\}\subseteq S$ *such that* $diam T^{k_n}(C_n\cup\{x\})\to 0, n\to+\infty$.

(iii) For any $n\in\mathbb{N}$, C_n *is a Kronecker set respect to* S.

(iv) K *is a chaotic set with respect to* S.

Proof. As (X,T) is a non-trivial $\mathcal{F}$-mixing system, X has no isolated points. Let $Y=\{y_1,y_2,\cdots\}$ be a countable dense subset of X and $Y_n=\{y_1,y_2,\cdots,y_n\}$. Let $\mathcal{F}'$ be the smallest family containing $\{N(U,V): U,V$ are opene sets of $X\}$. As (X,T) is $\mathcal{F}$-mixing, $\mathcal{F}'$ is a filter.

Let $\{O_n\}_{n=1}^{\infty}$ be a countable base of X, $a_0=0$ and $V_0=X$. We have the following claim.

Claim: There are $\{a_n\}\subseteq\mathbb{N}$, $\{k_n\}\subseteq S$, opene subsets $\{U_n\}_{n=1}^{\infty}$ and $\{V_{n,1},V_{n,2},\cdots,V_{n,a_n}\}_{n=1}^{\infty}$ of X such that:

(1) $2a_{n-1}\le a_n\le 2a_{n-1}+n$.

(2) $diam V_{n,i}<\frac{1}{n}$, $i=1,2,\ldots,a_n$.

(3) The closures $\{\overline{V_{n,i}}\}_{i=1}^{a_n}$ are pairwise disjoint.

(4) $\overline{V_{n,2i-1}}\cup\overline{V_{n,2i}}\subset V_{n-1,i}$, $i=1,2,\cdots,a_{n-1}$.

(5) $Y_n\subset B(\bigcup\limits_{i=1}^{a_n}V_{n,i},\frac{1}{n})$, where $B(A,\epsilon):=\{x\in X: d(x,A)<\epsilon\}$.

(6) $diam U_n<\frac{1}{n}$ and $S\cap N(x,U_n)\in k\mathcal{F}'$.

(7) $T^{k_n}x\in U_n$ and $T^{k_n}V_{n,j}\subseteq U_n$, $j=1,2,\cdots,a_n$.

(8) For any $\alpha\in\{1,2,\cdots,a_n\}^{a_n}$ there is $m(\alpha)\in S$ such that

$$T^{m(\alpha)}V_{n,i}\subseteq O_{\alpha(i)}, i=1,2,\cdots,a_n.$$

Proof of the Claim: For $n\in\mathbb{N}$, let $\{G_j^{(n)}\}_{j=1}^{b_n}$ be a finite open cover of X with $diam G_j^{(n)}<\frac{1}{n}, j=1,2,\cdots,b_n$ and $S\in k\mathcal{F}$. As $S=S\cap\mathbb{Z}_+=\bigcup\limits_{j=1}^{b_n}(S\cap N(x,G_j^{(n)}))$

and $S \in k\mathcal{F} \subseteq k\mathcal{F}'$, there is some $j_0 \in \{1, 2, \cdots, b_n\}$ such that $S \cap N(x, G_{j_0}^{(n)}) \in k\mathcal{F}'$ since $k\mathcal{F}'$ has Ramsey property. We set $U_n = G_{j_0}^{(n)}, n \in \mathbb{N}$. Then $\{U_n\}$ satisfies (6).

Let $a_1 = 1$ and $W_{1,1}$ be a neighborhood of y_1 with $diam W_{1,1} < 1$. As $N(W_{1,1}, U_1) \in \mathcal{F}'$ and $S \cap N(x, U_1) \in k\mathcal{F}'$ there is some $k_1 \in S \cap N(x, U_1) \cap N(W_{1,1}, U_1)$. It is clear that $T^{k_1} x \in U_1$. Moreover, there is an opene set $W'_{1,1} \subseteq W_{1,1}$ such that $T^{k_1} W'_{1,1} \subseteq U_1$. Since $N(W'_{1,1}, O_1) \cap S \neq \emptyset$, we can take some $m(1) \in N(W'_{1,1}, O_1) \cap S$. Then there is some opene set $V_{1,1} \subseteq W'_{1,1}$ such that $T^{m(1)} V_{1,1} \subseteq O_1$.

Assume for $1 \leq j \leq n-1$ we have $\{a_j\}_{j=1}^{n-1}$, $\{k_j\}_{j=1}^{n-1}$ and $\{V_{j,1}, V_{j,2}, \cdots, V_{j,a_j}\}_{j=1}^{n-1}$ satisfying conditions (1)-(8).

We take $2a_{n-1} \leq a_n \leq 2a_{n-1} + n$ and opene subsets $W_{n,1}, W_{n,2}, \cdots, W_{n,a_n}$ of X such that

(9) $diam W_{n,i} < \frac{1}{2n}$, $i = 1, 2, \cdots, a_n$.

(10) The closures $\{\overline{W_{n,i}}\}_{i=1}^{a_n}$ are pairwise disjoint.

(11) $\overline{W_{n,2i-1}} \cup \overline{W_{n,2i}} \subset V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.

(12) $Y_n \subset B(\bigcup_{i=1}^{a_n} W_{n,i}, \dfrac{1}{2n})$.

As $N(W_{n,i}, U_n) \in \mathcal{F}'$ for each $1 \leq i \leq a_n$, $\bigcap_{i=1}^{a_n} N(W_{n,i}, U_n) \in \mathcal{F}'$. And as $S \cap N(x, U_n) \in k\mathcal{F}'$ there is some $k_n \in S \cap N(x, U_n) \cap \bigcap_{i=1}^{a_n} N(W_{n,i}, U_n)$. Hence there are opene sets $W'_{n,i} \subseteq W_{n,i}$ such that

(13) $T^{k_n} W'_{n,i} \subseteq U_n$ for each $1 \leq i \leq a_n$.

We set $\{1, 2, \cdots, a_n\}^{a_n} = \{\alpha_i\}_{i=1}^{t_n}$, where $t_n = a_n^{a_n}$.

Since $N(\Pi_{i=1}^{a_n} W'_{n,i}, \Pi_{i=1}^{a_n} O_{\alpha_1(i)}) = \bigcap_{i=1}^{a_n} N(W'_{n,i}, O_{\alpha_1(i)}) \in \mathcal{F}'$, there is

$$m(\alpha_1) \in S \cap N(\Pi_{i=1}^{a_n} W'_{n,i}, \Pi_{i=1}^{a_n} O_{\alpha_1(i)}).$$

Then we can choose $V_{n,i}^{(1)} \subseteq W'_{n,i}$ such that

$$T^{m(\alpha_1)} V_{n,i}^{(1)} \subseteq O_{\alpha_1(i)},\ i = 1, 2, \cdots, a_n.$$

Take

$$m(\alpha_2) \in S \cap N(\Pi_{i=1}^{a_n} V_{n,i}^{(1)}, \Pi_{i=1}^{a_n} O_{\alpha_2(i)}).$$

Hence we can choose $V_{n,i}^{(2)} \subseteq V_{n,i}^{(1)}$ such that

$$T^{m(\alpha_2)} V_{n,i}^{(2)} \subseteq O_{\alpha_2(i)},\ i = 1, 2, \cdots, a_n.$$

Assume for $1 \le j \le t_n - 1$, we have $m(\alpha_1), m(\alpha_2), \cdots, m(\alpha_j) \in S$ and $W'_{n,i} \supseteq V_{n,i}^{(1)} \supseteq V_{n,i}^{(2)} \supseteq \cdots \supseteq V_{n,i}^{(j)}$ such that $T^{m(\alpha_h)} V_{n,i}^{(h)} \subseteq O_{\alpha_h(i)}$, $i = 1, 2, \cdots, a_n$, $h = 1, 2, \cdots, j$.

Take

$$m(\alpha_{j+1}) \in S \cap N(\Pi_{i=1}^{a_n} V_{n,i}^{(j)}, \Pi_{i=1}^{a_n} O_{\alpha_{j+1}(i)}).$$

Hence we can choose $V_{n,i}^{(j+1)} \subseteq V_{n,i}^{(j)}$ such that

$$T^{m(\alpha_{j+1})} V_{n,i}^{(j+1)} \subseteq O_{\alpha_{j+1}(i)},\ i = 1, 2, \cdots, a_n.$$

By induction we have $\{m(\alpha_j)\}_{j=1}^{t_n}$ and $\{V_{n,i}^{(j)}\}_{j=1}^{t_n}$. Let $V_{n,i} = V_{n,i}^{(t_n)}, i = 1, 2, \cdots, a_n$. Then (8) holds. **This ends the proof of the claim**.

Let $C_n = \bigcap_{j=n}^{\infty} \bigcup_{i=1}^{2^{j-n}a_n} \overline{V_{j,i}}$. Then $C_1 \subseteq C_2 \subseteq \cdots$. By (1)-(4) C_n is a Cantor set. By (2),(4) and (5), $K = \bigcup_{n=1}^{\infty} C_n$ is dense in X. By (7) we have $diam T^{k_n}(C_n \cup \{x\}) \to 0, n \to \infty$.

Now, using (8) we are going to prove (iii). Let $n \in \mathbb{N}$, $g \in \mathcal{C}(C_n, X)$ and $\epsilon > 0$. Since C_n is a compact set, there exists $m \in \mathbb{N}$ with $m \ge n$ such that if $x, y \in C_n$ and $d(x, y) \le \frac{1}{m}$, then $d(g(x), g(y)) < \frac{\epsilon}{2}$. For every $i \in \{1, 2, \cdots, 2^{m-n}a_n\}$, we choose $x_i \in C_n \cap \overline{V_{m,i}}$. Note that when $z_i \in \overline{V_{m,i}}$, one has $d(z_i, x_i) \le \frac{1}{m}$. So by the choosing of m, when $z_i \in \overline{V_{m,i}} \cap C_n$, one has $d(g(z_i), g(x_i)) < \frac{\epsilon}{2}$.

Since $\{O_n\}_{n=1}^{\infty}$ is a countable base of X, for every $i \in \{1, 2, \cdots, 2^{m-n}a_n\}$ there exist $n_i \in \mathbb{N}$ such that $g(x_i) \in O_{n_i}$ and $diam(O_{n_i}) < \frac{\epsilon}{2}$. Let $M = \max\{n_1, n_2, \cdots, n_{2^{m-n}a_n}, m\}$ and $i(j) = [\frac{j-1}{2^{M-m}}] + 1$ for $j \in \{1, 2, \cdots, 2^{M-n}a_n\}$. By (8), there exist $k = k(n, \epsilon) \in S$ such that $T^k V_{M,j} \subset O_{n_{i(j)}}$ for every $j \in \{1, 2, \cdots, 2^{M-n}a_n\}$.

For each $x \in C_n$, there exists $j \in \{1, 2, \cdots, 2^{M-n}a_n\}$ such that $x \in \overline{V_{M,j}}$. Thus $x \in \overline{V_{m,i(j)}}$, therefore $d(g(x), g(x_{i(j)})) < \frac{\epsilon}{2}$. Since $T^k x \in \overline{T^k V_{M,j}} \subset \overline{O}_{n_{i(j)}}$, one has $d(T^k x, g(x_{i(j)})) \le diam(\overline{O}_{n_{i(j)}}) \le \frac{\epsilon}{2}$. Combining the two facts above, one gets

$$d(g(x), T^k(x)) \le d(T^k x, g(x_{i(j)})) + d(g(x), g(x_{i(j)})) < \epsilon.$$

This shows that C_n is a Kronecker set with respect to S.

Finally, we show (iv). Set $g \in \mathcal{C}(K, X)$ and $g_n = g|_{C_n} \in C(C_n, X)$ for every $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, by (iii) there exists $q_i \in S$ such that $d(g_i(x), T^{q_i} x) < \frac{1}{i}$ for $x \in C_i$. As $C_1 \subset C_2 \subset C_3 \cdots$ and $K = \bigcup_{i=1}^{+\infty} C_n$, one has $\lim_{i \to +\infty} T^{q_i}(x) = g(x)$ for $x \in K$. $\square$

Remark: 1. A subset C of X is called a **Xiong-chaotic set with respect to** $S \subset \mathbb{Z}_+$ if for any subset A of C and for any continuous map $F : A \to X$ there is a subsequence $\{q_i\} \subset S$ such that $\lim_{i\to\infty} T^{q_i}(x) = F(x)$ for every $x \in A$ ([XY]). The authors showed if (X,T) is a dynamical system where X is a separable locally compact metric space containing at least two points, then (X,T) is weakly mixing if and only if there are some infinite set S and $c-$dense $F_\sigma-$subset C of X, which is chaotic with respect to S. In fact we can show that the K in Theorem 5.2 is a Xiong-chaotic set with respect to S and a proof is included in the appendix.

2. In [Ak1,Ak2] the author discussed the Kronecker set in weakly mixing systems. Our results tell more about Kronecker sets of weakly mixing systems and the proof presented here is totally different from [Ak1,Ak2].

Corollary 5.3. *Let (X, f) be a TDS and $\mathcal{F}$ be a full family. Then (X, f) is $\mathcal{F}$-mixing if and only if for any $S \in k\mathcal{F}$ there is a dense subset C of X, which is chaotic with respect to S.*

§6 Appendix

Recall a subset C of X is called a Xiong-chaotic set with respect to $S \subset \mathbb{Z}_+$ if for any subset A of C and for any continuous map $F : A \to X$ there is a subsequence $\{q_i\} \subset S$ such that $\lim_{i\to\infty} T^{q_i}(x) = F(x)$ for every $x \in A$. Xiong and Yang characterized weak mixing and strong mixing in terms of chaoticity. A similar characterization for $\mathcal{F}$-mixing systems was obtained in [SY] using Xiong-Yang's method. Now we gave another proof of the fact using the method developed in the previous section. Note that this proof is simpler than the one giving by Xiong and Yang [XY].

Theorem. *If (X,T) is a dynamical system where X is a separable locally compact metric space containing at least two points and $\mathcal{F}$ is a full family, then (X,T) is $\mathcal{F}$-mixing if and only if for any $S \in k\mathcal{F}$ there is $c-$dense $F_\sigma-$subset K of X which is Xiong-chaotic with respect to S.*

Proof. We show the necessity and the sufficiency is easy.

As X is a $\mathcal{F}-$mixing system with at least two points, there are no isolated points for X. Let $\{O_i\}$ be an open countable base of X. And in addition we can assume $\{diamO_i\}$ is a decreasing sequence which tends to zero. To see this, observe first that we have a base $\{O'_i\}$ such that $\overline{O'_i}$ is compact for each $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $A_i = O'_1 \cup \ldots \cup O'_i$ has a finite open cover $\mathcal{A}_i$ the diameter of each element of which is less than $\frac{1}{i}$. List the elements of $\mathcal{A}_i$ to form a sequence $\{O_i\}$, then $\{O_i\}$ is the one we need. By the proof of Theorem 5.2 we have the following fact ($a_0 = 0, V_0 = X$):

Fact: There are $\{a_n\} \subseteq \mathbb{N}$, $\{k_n\} \subseteq S$ and opene subsets $\{V_{n,1}, V_{n,2}, \cdots, V_{n,a_n}\}_{n=1}^{\infty}$ of X such that:

(1) $2a_{n-1} \leq a_n \leq 2a_{n-1} + n$.

(2) $diamV_{n,i} < \frac{1}{n}$, $i = 1, 2, \ldots, a_n$.

(3) The closures $\{\overline{V_{n,i}}\}_{i=1}^{a_n}$ are pairwise disjoint compact subsets of X.

(4) $\overline{V_{n,2i-1}} \cup \overline{V_{n,2i}} \subset V_{n-1,i}$, $i = 1, 2, \cdots, a_{n-1}$.

(5) $Y_n \subset B(\bigcup_{i=1}^{a_n} V_{n,i}, \frac{1}{n})$, where $B(A, \epsilon) := \{x \in X : d(x, A) < \epsilon\}$.

(6) For any $\alpha \in \{1, 2, \cdots, a_n\}^{a_n}$ there is $m(\alpha) \in S$ such that

$$T^{m(\alpha)} V_{n,i} \subseteq O_{\alpha(i)}, i = 1, 2, \cdots, a_n.$$

Let $C_n = \bigcap_{j=n}^{\infty} \bigcup_{i=1}^{2^{j-n} a_n} \overline{V_{j,i}}$. Then $C_1 \subseteq C_2 \subseteq \cdots$. And let $K = \bigcup_{n=1}^{\infty} C_n$.

Now we show that for any subset A of K and for any continuous map $F : A \to X$ there is a subsequence $\{q_i\} \subset S$ such that $\lim_{i\to\infty} T^{q_i}(x) = F(x)$ for every $x \in A$.

Let $A_n = \{x \in A :$ there is some $1 \le i \le n$ and $1 \le a_x \le a_n$ such that $x \in V_{n,a_x} \cap A \subseteq F^{-1}(O_i)\}$ (Note that A_n may be empty when n is small). It is easy to see that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ and $\bigcup_{n=1}^{+\infty} A_n = A$. If A_n is not empty, then let

$$\begin{aligned}&\{V_{n,j} : \exists x \in A_n \text{ and } 1 \le i \le n \text{ such that } x \in V_{n,j} \cap A \subset F^{-1}(O_i)\}\\ = &\{V_{n,i_1^{(n)}}, V_{n,i_2^{(n)}}, \cdots, V_{n,i_{b_n}^{(n)}}\}, \text{ where } 1 \le i_1^{(n)} < i_2^{(n)} < \cdots < i_{b_n}^{(n)} \le a_n.\end{aligned}$$

Let $\alpha_n \in \{1, 2, \cdots, a_n\}^{a_n}$ be any one with $\alpha_n(i_j^{(n)}) = \max\{1 \le k \le n : V_{n,i_j^{(n)}} \cap A \subseteq F^{-1}(O_k)\}$, $1 \le j \le b_n$. And let $q_n = m(\alpha_n)$.

Now we show $\lim_{i\to\infty} T^{q_i}(x) = F(x)$ for every $x \in A$.

Let $\epsilon > 0$ and there is some $N \in \mathbb{N}$ such that $diam O_n < \epsilon$ when $n > N$. Fix $x \in A$. Take $t > N$ such that O_t is a neighborhood of $F(x)$. As F is continuous, $F^{-1}(O_t)$ is an open neighborhood of x in A. Thus there is some $n_t > t$ and $1 \le a_x \le a_{n_t}$ such that $x \in V_{n_t,a_x} \cap A \subseteq F^{-1}(O_t)$ by (2). By (4), for each $j \in \mathbb{N}$ there is some $1 \le a_x^j \le a_{n_t+j}$ such that

$$x \in V_{n_t+j,a_x^j} \cap A \subseteq V_{n_t,a_x} \cap A \subseteq F^{-1}(O_t). \tag{7}$$

Thus $\alpha_{n_t+j}(a_x^j) \ge t > N$ for each $j \in \mathbb{N}$. Moreover, by the definition of $\{q_n\}$ we have for any $j \in \mathbb{N}$

$$T^{q_{n_t+j}} V_{n_t+j,a_x^j} = T^{m(\alpha_{n_t+j})} V_{n_t+j,a_x^j} \subseteq O_{\alpha_{n_t+j}(a_x^j)}, \tag{8}$$

and

$$x \in V_{n_t+j,a_x^j} \cap A \subseteq F^{-1}(O_{\alpha_{n_t+j}(a_x^j)}). \tag{9}$$

By (9) we have $F(x) \in O_{\alpha_{n_t+j}(a_x^j)} \ne \emptyset$. By (8) $T^{q_{n_t+j}} x \in O_{\alpha_{n_t+j}(a_x^j)}$. Hence we have for each $j \in \mathbb{N}$

$$d(T^{q_{n_t+j}} x, F(x)) \le diam(O_{\alpha_{n_t+j}(a_x^j)}) < \epsilon.$$

That is $\lim_{i\to\infty} T^{q_i}(x) = F(x)$. □

References

[AG] E. Akin and E. Glasner, *Residual properties and almost equicontinuity*, J. d'Anal. Math. **84** (2001), 243-286.

[Ak1] E. Akin, *Recurrence in topological dynamical systems: Furstenberg families and Ellis actions*, Plenum Press, New York, 1997.

[Ak2] E. Akin, *Cantor sets and Mycielski sets for dynamical systems*, preprint (2003).

[AK] E. Akin and S. Kolyada, *Li-Yorke Sensitivity*, Nonlinearity **16** (2003), 1421–1433.

[Au1] J. Auslander, *On the proximal relation in topological dynamics*, Proc. Amer. Math. Soc. **11** (1960), 890-895.

[Au2] J. Auslander, *Minimal flows with a closed proximal cell*, Ergod. Th. and Dynam. Sys. **21** (2001), 641-645.

[Au3] J. Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies, 153. Amsterdam, 1988.

[BHM] F. Blanchard, B. Host and A. Maass, *Topological complexity*, Ergod. Th. and Dynam. Sys. **20** (2000), 641-662.

[E] R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, Inc., New York, 1969.

[F1] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, Math. Systems Theory **1** (1967), 1-49.

[F2] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, 1981.

[FK] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Analyse Math. **45** (1985), 117–168.

[H] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Combinatorial Theory Ser. A **17** (1974), 1–11.

[HY1] W. Huang, and X.D. Ye, *Devaney's chaos or 2-scattering implies Li-Yorke's chaos*, Topology Appl. **117 no.3** (2002), 259–272.

[HY2] W. Huang and X.D. Ye, *Topological complexity,return times and weak disjointness*, preprint.

[KR] H. B. Keynes and J. B. Robertson, *Eigenvalue theorems in topological transformation groups*, Trans. Amer. Math. Soc. **139** (1969), 359-369.

[M] J. H. Mai, *Devaney's Chaos implies existence of s-Scrambled sets*, preprint (2003).

[P] R. Peleg, *Weakly disjointness of transformation groups*, Proc. Amer. Math. Soc. **33** (1972), 165-170.

[SY] S. Shao and X. Ye, *$\mathcal{F}$-mixing and weak disjointness,*, Topology Appli, to appear.

[V] W. A. Veech, *Point-distal flows*, Amer.J.Math **92** (1970), 205-242.

[XY] J. Xiong and Z. Yang, *Chaos caused by a topologically mixing map*, Advanced Series in Dynamical Systems **9 World Scientific** (1990), 550-572.

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