SPACES OF ω-LIMIT SETS OF GRAPH MAPS

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Abstract. Let (X, f) be a dynamical system. In general the set of all ω-limit sets of f is not closed in hyperspace. In this paper we study the case when X is a graph, and show that the family of ω-limit sets of a graph map is closed with respect to the Hausdorff metric.

1. Introduction

A *dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and f is a continuous map from X to itself. For $x \in X$, $\{x, f(x), f^2(x), \ldots\}$ is called the *orbit* of x and denoted by $O(x, f)$. x is a *periodic point* if $f^n(x) = x$ for some $n \in \mathbb{N}$, and n is called the *period* of x. If $n = 1$, then x is also called a *fixed point* of f. A system is *transitive* if there exists some dense orbit. Denote the set of all limit points of an orbit $O(x)$ by $\omega(x, f)$ and call it the *ω-limit set* of f. For any $x \in X$, $\omega(x, f)$ is closed nonempty subset of X and it is strongly invariant (i.e. $f(\omega(x, f)) = \omega(x, f)$). Write $\mathbb{X}(f, \omega) = \{\omega(x, f) : x \in X\}$.

Omega limit sets give fundamental information about asymptotic behavior of a dynamical system. One of the basic tasks is to give a topological characterization of them. This task is very complicated even in the simplest one-dimensional case—the compact interval ([1], [2], [3], [4], [6]). Let I be a closed interval in $\mathbb{R}$ and let $f : I \to I$ be a continuous map. Then for any $x \in I$, $\omega(x, f)$ is (i) a periodic orbit, or (ii) an infinite compact nowhere dense set, or (iii) a finite union of connected subintervals which forms a periodic orbit ([3], [6]). Conversely, whenever $A \subseteq I$ is of one of the above forms then there is a continuous map $f : I \to I$ such that A is an ω-limit set of f. This result is generalized to a graph map in [7]. Another related problem is, for a given f and a closed strongly f-invariant set A, to find a condition in order to verify whether A is an ω-set of f or not. One can find these kinds of characterizations of ω-sets in [4] and [2].

Let I be a closed interval in $\mathbb{R}$ and let $f : I \to I$ be a continuous map. The map $\omega : I \mapsto \omega(x, f)$ was studied in [5] and it was shown that this

2000 *Mathematics Subject Classification.* 37E25 (primary), 54B20 (secondary).

Key words and phrases. ω-limit set, graph map, hyperspace.

The first author is supported by the Special Foundation of National Prior Basis Research of China (Grant No. G1999075108) and the second author is supported by National Natural Science Foundation of China (10501042) and Natural Science Foundation of AnHui (050460101).

map is far from continuous. Hence it is somewhat surprising that the image of this map endowed with the Hausdorff metric is closed. This was proved in [4] and the proof of this result is rather long but quite elementary and ingenious. In a similar way, this result was extended to circle maps in [9]. In this paper we show that the family of ω-limit sets of a graph map is closed with respect to the Hausdorff metric. The proof we offer is different from [4], [9] and simpler. Also in the proof we give a characterization of ω-limit sets of a graph map.

2. Preliminary

In the article, integers, nonnegative integers, natural numbers, real numbers and the complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

Let (X, d) be a compact metric space. The *hyperspace* $\mathbb{X}$ is the set of all nonempty closed subsets of X. The Hausdoff metric d_H on $\mathbb{X}$ is defined as follows:

$$d_H(V, W) = \max\{d(v, W), d(w, V) : v \in V, w \in W\}, \ \forall\, V, W \in \mathbb{X},$$

where $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ for $x \in X$ and $Y \subseteq X$. It is well known that $(\mathbb{X}, d_H)$ is a compact metric space (see [8], for example).

Let (X, f) be a dynamical system. Recall that a subset $A \subseteq X$ is f-*invariant* if $f(A) \subseteq A$, and *strongly f-invariant* if $f(A) = A$. Let $\mathbb{X}_1(f)$ ($\mathbb{X}_2(f)$) be the set of all nonempty (strongly) f-invariant closed subsets. Obviously one has that $\mathbb{X}(f, \omega) \subseteq \mathbb{X}_2(f) \subseteq \mathbb{X}_1(f)$. And it is easy to verify the following proposition.

Proposition 2.1. *$\mathbb{X}_1(f)$ and $\mathbb{X}_2(f)$ are closed subspaces of $(\mathbb{X}, d_H)$. Especially, both are compact.*

But generally the space $\mathbb{X}(f, \omega)$ need not be closed in $\mathbb{X}$.

Example 2.2. Let $D = \{re^{i\theta} \in \mathbb{C} : 0 \leq r \leq 1, \theta \in \mathbb{R}\}$ be the unit disc and $f : D \to D, re^{i\theta} \mapsto re^{i(\theta+r)}$. Then (D, f) is a dynamical system. It is easy to verify $\mathbb{D}(f, \omega)$ is not closed in $\mathbb{D}$.

In the next section it will be shown that when X is a graph, $\mathbb{X}(f, \omega)$ is closed. Now recall some definitions about a graph. By a *graph* one means a connected compact one-dimensional polyhedron in $\mathbb{R}^3$. A continuous map from a graph to itself is called a *graph map*. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. Then a graph G is a continuum (i.e. a nonempty, compact, connected metric space) which can be written as the union of finitely many arcs and any two of which are either disjoint or intersect only in one or both of their endpoints. Each of these arcs is called an *edge* of G, and its end is called a *vertex*. Since G is a polyhedron in $\mathbb{R}^3$, there are at least three edges in any circle of G. For a given graph G, a *subgraph* of G is a subset of G which is a graph itself. The *valence* of a vertex x is the number of edges that are incident on x, and if the number is n then one writes $val(x) = n$. A vertex of valence 1 is also called an *end* of G, and a vertex x with $val(x) \geq 3$ is said a *branching point* of G. The set of branching points of G is denoted by $Br(G)$. A *tree* is a graph without any subset which is homeomorphic to the unit circle. A *star* is either a tree having only one branching point or an arc.

For convenience, we assume that the length of every edge of G is greater than 1. Hence any non-degenerate connected closed subset of G with diameter less than 1 is a star. Let $x, y \in G$. The arc with ends $\{x, y\}$ is denoted by $[x, y]$ or $[y, x]$. Write $(a, b) = [a, b] \setminus \{a, b\}$, and similarly one defines $[a, b)$ and $(a, b]$. $[x; y]$ is also used to denote an arc with ends $\{x, y\}$, but in this case one means this arc starts from the point x and ends with y.

For a topological space X, the closure of a subset $A \subseteq X$ is denoted by $\overline{A}$. When (X, d) is a metric space, one writes $B(x, \varepsilon)$ for the ε-ball $\{x' \in X : d(x, x') < \varepsilon\}$ and $B(Y, \varepsilon) = \{x \in X : d(x, Y) < \varepsilon\}$, where $x \in X$, $Y \subseteq X$ and $\varepsilon > 0$.

3. Spaces of ω-limit sets of graph maps

The following theorem is the main result of this paper.

Theorem 3.1. *Let G be a graph and let $f : G \to G$ be a continuous map. Then the set of all ω-limit sets endowed with Hausdoff metric is compact.*

Before proving the theorem, one needs some notations and lemmas. Recall that $\mathbb{G}$ is the hyperspace of G and $\mathbb{G}(f, \omega)$ is the set of ω-limit sets of G. Let $v_1, v_2, \ldots$ be an infinite sequence in G. For any $n \in \mathbb{N}$ write

$$V_n = O(v_n, f), V = \bigcup_{n=1}^{\infty} V_n, X_n = \omega(v_n, f) \text{ and } X = \bigcup_{n=1}^{\infty} X_n.$$

Assume that the sequence $\{X_n\}_{n=1}^{\infty}$ converges to W in $(\mathbb{G}, d_H)$. To prove $\mathbb{G}(f, \omega)$ is closed, it only needs to show $W \in \mathbb{G}(f, \omega)$.

Now let's sketch the idea of the proof. Firstly we reduce the system to the case satisfying Condition I-III. The main reason is to exclude the easier case when $(W, f|_W)$ is transitive. Then we study the system under Condition I-III. Lemma 3.5 gives a condition under which W belongs to $\mathbb{G}(f, \omega)$. The rest part is to show that the condition of this lemma can be satisfied.

If there are infinitely many elements of $\{X_n\}_{n=1}^{\infty}$ which are equal, then it is easy to see $W \in \mathbb{G}(f, \omega)$. So we assume:

Condition I : For any $n, m \in \mathbb{N}$ with $n \neq m$, $X_n \neq X_m$ and $d_H(X_n, W) < 2^{-n-1}$.

Observe that if $V_n \cap V_m \neq \emptyset$ then $X_n = X_m$ for $n, m \in \mathbb{N}$. So in addition one can assume the following condition holds:

Condition II : For any $n, m \in \mathbb{N}$ with $n \neq m$ one has $V_n \cap V_m = \emptyset$ and

$$d_H(\overline{V_n}, X_n) < 2^{-n-1}, \quad d_H(\overline{V_n}, W) < 2^{-n}.$$

Lemma 3.2. *Assume that conditions I,II hold. If for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $n \geq N, y \in V_n$ and $w \in W$ such that*

$$\sup\{d(f^i(y), f^i(w)) : i \in \mathbb{Z}_+\} \leq \varepsilon,$$

then $(W, f|_W)$ is transitive and hence $W \in \mathbb{G}(f, \omega)$.

Proof. Let $\delta_0 = 1/8$. By the assumption, there are $n_1 \geq 1$, $y_1 \in V_{n_1}$ and $w_1 \in W$ such that $d_H(\overline{V_{n_1}}, X_{n_1}) < \delta_0$, $d_H(X_{n_1}, W) < \delta_0$ and

$$\sup\{d(f^i(y_1), f^i(w_1)) : i \in \mathbb{Z}_+\} \leq \delta_0.$$

Choose $k_1 \in \mathbb{N}$ such that $d_H(O_{k_1}(y_1, f), \overline{O(y_1, f)}) < \delta_0$, where $O_k(x, f) = \{f^j(x) : 0 \leq j \leq k\}$ for $x \in G$ and $k \in \mathbb{N}$. There is some $\delta_1 \in (0, \delta_0/8]$ such that for any $x \in B(w_1, 3\delta_1)$, one has

$$\sup\{d(f^i(x), f^i(w_1)) : i = 0, 1, \ldots, k_1\} < \delta_0.$$

Then for any $x \in B(w_1, 3\delta_1)$, one has

$$\begin{aligned} d_H(O_{k_1}(x, f), W) &< \delta_0 + d_H(O_{k_1}(w_1, f), W) \\ &\leq 2\delta_0 + d_H(O_{k_1}(y_1, f), W) \\ &< 3\delta_0 + d_H(\overline{O(y_1, f)}, W) \\ &\leq 3\delta_0 + d_H(\overline{O(y_1, f)}, X_{n_1}) + d_H(X_{n_1}, W) \\ &\leq 3\delta_0 + d_H(\overline{V_{n_1}}, X_{n_1}) + d_H(X_{n_1}, W) \\ &< 5\delta_0. \end{aligned} \tag{3.1}$$

By assumption, there are $n_2 > n_1$, $y_2' \in V_{n_2}$ and $w_2' \in W$ such that $d_H(\overline{V_{n_2}}, X_{n_2}) < \delta_1$, $d_H(X_{n_2}, W) < \delta_1$ and

$$\sup\{d(f^i(y_2'), f^i(w_2')) : i \in \mathbb{Z}_+\} \leq \delta_1.$$

Take $x_2 \in X_{n_2}$ such that $d(w_1, x_2) = d(w_1, X_{n_2}) \leq d_H(W, X_{n_2}) < \delta_1$ and take $j_2 \in \mathbb{Z}_+$ such that $d(f^{j_2}(y_2'), x_2) < \delta_1 - d_H(W, X_{n_2})$. Then $f^{j_2}(y_2') \in B(w_1, \delta_1)$ and $f^{j_2}(w_2') \in B(w_1, 2\delta_1)$. Let $y_2 = f^{j_2}(y_2')$ and $w_2 = f^{j_2}(w_2')$. Choose $k_2 > k_1$ such that $d_H(O_{k_2}(y_2, f), \overline{O(y_2, f)}) < \delta_1$. There is $\delta_2 \in (0, \delta_1/8]$ such that for any $x \in B(w_2, 3\delta_2)$, one has

$$\sup\{d(f^i(x), f^i(w_2)) : i = 0, 1, \ldots, k_2\} < \delta_1.$$

Then for any $x \in B(w_2, 3\delta_2)$, similar to (3.1) one gets $d_H(O_{k_2}(x, f), W) < 5\delta_1$.

Inductively, we have points $w_1, w_2, \ldots$ in W, positive integers $k_1 < k_2 < k_3 < \ldots$ and positive numbers $\delta_0 = 1/8 > \delta_1 > \delta_2 \ldots$ such that for any $n \in \mathbb{N}$, $\delta_n \le \delta_{n-1}/8$, $w_{n+1} \in B(w_n, 2\delta_n)$ and

$$d_H(O_{k_n}(x, f), W) < 5\delta_{n-1}, \ \forall x \in B(w_n, 3\delta_n).$$

Hence it is easy to see that the sequence $\{w_n\}$ converges to some point $w \in W$ and for any $n \in \mathbb{N}$, $w \in B(w_n, 3\delta_n)$ and

$$B(O(w, f), 5\delta_{n-1}) \supseteq B(O_{k_n}(w, f), 5\delta_{n-1}) \supseteq W.$$

Thus $O(w, f)$ is dense in W and $f|_W$ is transitive. The proof of the lemma is completed. □

Note that Lemma 3.2 holds for any compact space, not only for graph maps. By Lemma 3.2, to show $W \in \mathbb{G}(f, \omega)$ one only need consider the case when $(W, f|_W)$ is not transitive. Hence by Lemma 3.2, one can assume:

Condition III : There exists $\varepsilon_0 \in (0, 1/2]$ such that for any $y \in V$ and $w \in W$, one has

$$\sup\{d(f^i(y), f^i(w)) : i \in \mathbb{Z}_+\} > \varepsilon_0.$$

Obviously, if Condition III holds, then $V \cap W = \emptyset$.

In the sequel we always assume that Condition I-III hold. Fix ε_0 defined in Condition III and $\varepsilon \in (0, \varepsilon_0/2]$ such that $f(B(x, \varepsilon)) \subseteq B(f(x), \varepsilon_0)$ for all $x \in G$. Let $\mathbb{Y}(\varepsilon)$ be the set of all non-degenerate connected closed subsets of G contained in $B(W, \varepsilon)$ and with diameter less than ε. By our assumption of G any element of $\mathbb{Y}(\varepsilon)$ is a star.

Definition 3.3. Let $Y, Y' \in \mathbb{Y}(\varepsilon)$. If there exists a finite set $\{Y_0, Y_1, \ldots, Y_n\} \subseteq \mathbb{Y}(\varepsilon)$, $n \in \mathbb{N}$ such that $Y = Y_0$, $Y' = Y_n$ and $f(Y_{i-1}) \supseteq Y_i$ for any $i = 1, 2, \ldots, n$, then one denotes it by $Y \xrightarrow{(f,\varepsilon)} Y'$.

It is easy to verify the following lemma

Lemma 3.4. *(1) " $\xrightarrow{(f,\varepsilon)}$ " is a transitive relation on the set $\mathbb{Y}(\varepsilon)$.*

(2) If $Y \xrightarrow{(f,\varepsilon)} Y'$, then there exist a connected closed subset Z and $n \in \mathbb{N}$ such that $f^n(Z) = Y'$ and $\bigcup_{i=0}^n f^i(Z) \subseteq B(W, \varepsilon)$.

The following lemma offers a condition under which a set can belong to $\mathbb{G}(f, \omega)$.

Lemma 3.5. *Let $\varepsilon_0/2 \ge \delta_1 \ge \delta_2 \ge \delta_3 \ge \ldots$ be a sequence of positive numbers with $\lim_{i\to\infty} \delta_i = 0$ and let $\{Y_i\}_{i=1}^\infty$ be a sequence of non-degenerate connected closed subsets of G. If for any $i \in \mathbb{N}$, $Y_i \in \mathbb{Y}(\delta_i)$, $Y_i \xrightarrow{(f,\delta_i)} Y_{i+1}$ and $W \subseteq B(\bigcup_{j=i}^\infty Y_j, \delta_i)$, then $W \in \mathbb{G}(f, \omega)$.*

Proof. For any $i \in \mathbb{N}$, by Lemma 3.4-(2) there exist a connected closed subsets Z_i of Y_i and $n_i \in \mathbb{N}$ such that

$$f^{n_i}(Z_i) = Y_{i+1} \text{ and } \bigcup_{k=0}^{n_i} f^k(Z_i) \subseteq B(W, \delta_i).$$

Let $m_i = n_1 + n_2 + \ldots + n_i$ $(m_0 = 0)$, then $\bigcap_{i=1}^{\infty} f^{-m_{i-1}}(Z_i) \neq \emptyset$ and let $x \in \bigcap_{i=1}^{\infty} f^{-m_{i-1}}(Z_i)$. Since $O(f^{m_i}(x), f) \subseteq B(W, \delta_{i+1})$, one has $\omega(x, f) \subseteq W$.

On the other hand, since $\bigcup_{j=i+1}^{\infty} Y_j \subseteq B(O(f^{m_i}(x), f), \delta_{i+1})$, on has

$$W \subseteq B(\bigcup_{j=i+1}^{\infty} Y_j, \delta_{i+1}) \subseteq B(O(f^{m_i}(x), f), 2\delta_{i+1}).$$

Hence $W \subseteq \omega(x, f)$. Thus we have $W = \omega(x, f) \in \mathbb{G}(f, \omega)$. □

Definition 3.6. Let $Y \in \mathbb{Y}(\varepsilon)$. If $\{i \in \mathbb{N} : Y \cap V_i \neq \emptyset\}$ is infinite, then Y is called a (P, ε)-*star*. Let $A = [w; y] \in \mathbb{Y}(\varepsilon)$ be an arc with $w \in W$ and $y \in V$. If for any $x \in (w, y]$ and $n \in \mathbb{N}$ there is some $i \geq n$ such that $V_i \cap (w, x] \neq \emptyset$, then A is called a (P, ε)-*arc*.

Denote the set of all (P, ε)-star and the set of all (P, ε)-arc by $\mathbb{Y}(P, \varepsilon)$ and $\mathbb{A}(P, \varepsilon)$ respectively.

By the definition one gets the following lemma readily.

Lemma 3.7. *(1)* $\mathbb{A}(P, \varepsilon) \subseteq \mathbb{Y}(P, \varepsilon)$.

(2) For any $w \in W$, *there exists a point* $y \in B(w, \varepsilon) \cap V$ *such that* $[w; y]$ *is a* (P, ε)*-arc.*

(3) For any $Y \in \mathbb{Y}(P, \varepsilon)$ *and* $\varepsilon' \in (0, \varepsilon]$, *there exists a* (P, ε')*-arc* A *with* $A \subseteq Y$.

Let $Y \in \mathbb{Y}(P, \varepsilon)$, set

$$\text{(3.2)} \quad \mathbb{Y}(\varepsilon, Y) = \{ Y' \in \mathbb{Y}(\varepsilon) : Y \xrightarrow{(f,\varepsilon)} Y' \text{ and there exists } Y'' \in \mathbb{Y}(P, \varepsilon) \text{ such that } Y' = Y'' \text{ or } Y' \xrightarrow{(f,\varepsilon)} Y''\}.$$

Lemma 3.8. *(1) For any* $Y \in \mathbb{Y}(P, \varepsilon)$, $\mathbb{Y}(\varepsilon, Y) \neq \emptyset$.

(2) If $Y, Y' \in \mathbb{Y}(P, \varepsilon)$ *and* $Y \xrightarrow{(f,\varepsilon)} Y'$, *then* $\mathbb{Y}(\varepsilon, Y') \subseteq \mathbb{Y}(\varepsilon, Y)$.

(3) If $Y' \in \mathbb{Y}(P, \varepsilon)$ *and* $Y \in \mathbb{Y}(\varepsilon, Y')$ *with* $diam f(Y) < \varepsilon$, *then* $f(Y) \in \mathbb{Y}(\varepsilon, Y')$.

Proof. (1) and (2) are easy to be verified. We now prove (3). Since $diam f(Y) < \varepsilon$, one has $Y' \xrightarrow{(f,\varepsilon)} Y \xrightarrow{(f,\varepsilon)} f(Y)$. If $f(Y)$ is a (P, ε)-star, then by the definition $f(Y) \in \mathbb{Y}(\varepsilon, Y')$. Now assume that $f(Y)$ is not a (P, ε)-star. By the definition of $\mathbb{Y}(\varepsilon, Y')$, there is some (P, ε)-star Y'' such that $Y' \xrightarrow{(f,\varepsilon)} Y \xrightarrow{(f,\varepsilon)} Y''$. Since $f(Y)$ is not a (P, ε)-star, $f(Y) \neq Y''$. Hence by the definition of $Y \xrightarrow{(f,\varepsilon)} Y''$ it is easy to see that $f(Y) \xrightarrow{(f,\varepsilon)} Y''$. Hence $f(Y) \in \mathbb{Y}(\varepsilon, Y')$. □

Lemma 3.9. *Let $A = [w; y]$ be a (P, ε)-arc and w a periodic point. Then there exist $x \in (w, y)$ and a (P, ε)-star Y such that $[x, y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. Let m be the period of w and $val(w) = k$. Take positive numbers $\delta_{-1} > \delta_0 > \delta_1 > \ldots > \delta_{2k}$ such that $\delta_{-1} < \min\{d(w, y), \varepsilon_0\}$, $(B(w, \delta_{-1}) \setminus \{w\}) \cap Br(G) = \emptyset$ and for any $i = 0, 1, 2, \ldots, 2k$, $j = 1, 2, \ldots, m$, one has

$$f^j(B(w, \delta_i)) \subseteq B(f^j(w), \delta_{i-1}). \tag{3.3}$$

Let $\delta = \min\{\delta_{2k}, \delta_{i-1} - \delta_i : i = 0, 1, \ldots, 2k\}/2$. By Condition II and the definition of (P, ε)-arc, there is some $N \in \mathbb{N}$ such that

$$d_H(\overline{V_N}, W) < \delta \text{ and } A \cap V_N \cap B(w, \delta) \neq \emptyset. \tag{3.4}$$

Take any point $z \in A \cap V_N \cap B(w, \delta)$. By Condition III and (3.3), there are positive integers $n_0 < n_1 < n_2 < \ldots < n_{2k}$ such that for any $i = 0, 1, \ldots, 2k$, one has $f^{mn_i}(z) \in B(w, \delta_{2k-i-1}) \setminus B(w, \delta_{2k-i})$ and $\{f^{mj}(z) : j = 0, 1, \ldots, n_i - 1\} \subseteq B(w, \delta_{2k-i})$. Hence there are integers $0 \leq p < q < r \leq 2k$ such that $f^{mn_p}(z), f^{mn_q}(z)$ and $f^{mn_r}(z)$ are in the same connected component of $B(w, \delta_{-1}) \setminus \{w\}$. Take $x \in (w, y)$ such that

$$\{f^{mj}(x) : j = 0, 1, 2 \ldots, n_r\} \subseteq B(w, \delta).$$

Let $Y = [f^{mn_r}(x), f^{mn_r}(z)]$, then $[x, y] \supseteq [x, z] \xrightarrow{(f,\varepsilon)} Y$. By (3.4), there is some $w' \in W$ such that $w' \in B(f^{mn_q}(z), \delta) \subseteq [w, f^{mn_r}(z)) \setminus B(w, \delta) \subseteq Y$. By Lemma 3.7-(2), $\overline{B(f^{mn_p}(z), \delta)}$ is a (P, ε)-star and hence Y is also a (P, ε)-star. □

Corollary 3.10. *Let $A = [w; y]$ be a (P, ε)-arc. If there is some $n \in \mathbb{N}$ such that $f^n(w)$ is a periodic point, then there exist $x \in (w, y)$ and a (P, ε)-star Y such that $[x, y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. It is obvious that there exists $z \in (w, y] \cap V$ such that $[f^n(w); f^n(z)]$ is a (P, ε)-arc and $diam f^i([w, z]) < \varepsilon$ for any $i = 1, 2, \ldots, n$. According to Lemma 3.9, there exist $x' \in (f^n(w), f^n(z))$ and a (P, ε)-star Y such that $[x', f^n(z)] \xrightarrow{(f,\varepsilon)} Y$. Let $x \in f^{-n}(x') \cap (w, z)$, then one has

$$[x, y] \supseteq [x, z] \xrightarrow{(f,\varepsilon)} [x', f^n(z)] \xrightarrow{(f,\varepsilon)} Y.$$

□

Lemma 3.11. *Let $A = [w; y]$ be a (P, ε)-arc. Then there exist $x \in (w, y)$ and a (P, ε)-star Y such that $[x, y] \xrightarrow{(f,\varepsilon)} Y$.*

Proof. According to Corollary 3.10, it suffices to consider the case when $O(w, f)$ is infinite. By the definition of (P, ε)-arc, there are positive integers $k_1 < k_2 < \ldots$ and points $\{x_1, x_2, x_3, \ldots\} \subseteq (w, y]$ such that $x_i \in V_{k_i}$ and

$d(x_{i+1}, w) < d(x_i, w)/2$ for all $i \in \mathbb{N}$. By Condition III for every $i \in \mathbb{N}$ there is a unique $n_i \in \mathbb{N}$ such that $\varepsilon/2 \le d(f^{n_i}(x_i), f^{n_i}(w)) < \varepsilon_0$ and

$$d(f^n(x_i), f^n(w)) < \varepsilon/2, \ \forall n = 0, 1, \ldots, n_i - 1.$$

Obviously, one has $\lim_{i\to\infty} n_i = \infty$. Without loss of generality, one can assume $n_1 < n_2 < n_3 < \ldots$. Let $w_i = f^{n_i}(w)$, then the elements in $\{w_i\}_{i=1}^{\infty}$ are mutually distinct. Passing to a subsequence if necessary one can assume that $\lim_{i\to\infty} w_i = w'$ and for any $i \in \mathbb{N}$, one has

$$d(w_{i+1}, w') < d(w_i, w')/2 \text{ and } w_i \in [w_1, w').$$

Dropping the first finitely many points if necessary one can assume in addition that

$$d(w_1, w') < \varepsilon/4 \text{ and } [w_1, w') \cap Br(G) = \emptyset.$$

Take $\delta > 0$ such that $\delta < d(x_4, w)$ and for any $n = 0, 1, \ldots, n_4$,

$$f^n(B(w, \delta)) \subseteq B(f^n(w), d(w_5, w_4)/2).$$

Then for any $x \in B(w, \delta) \cap (w, y]$, one has $f^{n_4}(x) \in (w_5, w_3)$. Since $d(f^{n_4}(x_4), f^{n_4}(w)) = d(f^{n_4}(x_4), w_4) > \varepsilon/2$ and $d(w_1, w') < \varepsilon/4$, we have

$$[x, y] \supseteq [x, x_4] \xrightarrow{(f,\varepsilon)} [w', w_5] \text{ or } [x, y] \supseteq [x, x_4] \xrightarrow{(f,\varepsilon)} [w_3, w_1].$$

As $w_6 \in W \cap (w', w_5)$ and $w_2 \in W \cap (w_3, w_1)$, by Lemma 3.7-(2) $[w', w_5]$ and $[w_3, w_1]$ both are (P, ε)-arcs. Thus $[x, y] \xrightarrow{(f,\varepsilon)} Y$ holds for $Y = [w', w_5]$ or $Y = [w_3, w_1]$. This completes the proof of the lemma. □

Let $Y \in \mathbb{Y}(P, \varepsilon)$, and write

$$U(P, \varepsilon, Y) = \bigcup \{Y' : Y' \in \mathbb{Y}(\varepsilon, Y)\}. \tag{3.5}$$

Lemma 3.12. *Let $A = [w; y]$ be a (P, ε)-arc. Then $W \subseteq \overline{U(P, \varepsilon, A)}$.*

Proof. Choose $\varepsilon_1 \in (0, \varepsilon/2]$ such that $f(B(x, \varepsilon_1)) \subseteq B(f(x), \varepsilon/2)$ for all $x \in G$. Take any point $v \in V \cap B(w, \varepsilon_1) \cap A$, then $[w; v]$ and $f([w; v])$ are also (P, ε)-arcs and $f([w; v]) \in \mathbb{Y}(\varepsilon, A)$. Suppose that Lemma 3.12 does not hold, then there are $w_1 \in W$ and $\delta \in (0, \varepsilon_1]$ such that $B(w_1, \delta) \cap U(P, \varepsilon, A) = \emptyset$.

For a subset Z of G define $N(Z \cap V) = \{i \in \mathbb{N} : V_i \cap Z \neq \emptyset\}$ and $N_1(Z \cap X) = \{i \in \mathbb{N} : X_i \cap X \neq \emptyset\}$. Since $d_H(X_i, W) \to 0$ as $i \to \infty$, $N_1(B(w_1, \delta) \cap X)$ is cofinite. Hence $\mathbb{M} = N([w; v] \cap V) \cap N_1(B(w_1, \delta) \cap X)$ is an infinite subset of $\mathbb{N}$. For any $i \in \mathbb{M}$, one chooses a point $x_i \in V_i \cap [w, v]$. Since $f(x_i) \in f([w; v]) \subseteq U(P, \varepsilon, A)$ and $O(x_i, f) \cap B(w_1, \delta) \neq \emptyset$, there exists $y_i \in O(f(x_i), f) \cap U(P, \varepsilon, A)$ such that $f(y_i) \notin U(P, \varepsilon, A)$.

As $y_i \in U(P, \varepsilon, A)$, there is some $Y_i \in \mathbb{Y}(\varepsilon, A)$ such that $y_i \in Y_i$. By Lemma 3.8-(3), $diam Y_i \ge \varepsilon_1$. Let $i_1 < i_2 < \ldots$ be a sequence in $\mathbb{M}$ such that $\lim_{j\to\infty} y_{i_j} = y'$, $d(y_{i_1}, y') < \varepsilon_1, [y_{i_1}, y') \cap Br(G) = \emptyset, y_{i_k} \in [y_{i_1}, y')$ and $d(y_{i_{k+1}}, y') < d(y_{i_k}, y')/2, \forall k \in \mathbb{N}$.

It is obvious that $y' \in W$. If there exists some $k \in \mathbb{N}$ such that $y' \in Y_{i_k}$, then $[y_{i_k}, y']$ is a (P, ε_1)-arc contained in Y_{i_k} and hence $f([y_{i_k}, y'])$ is a (P, ε)-star. Thus we have $f([y_{i_k}, y']) \in \mathbb{Y}(\varepsilon, A)$ and $f(y_{i_k}) \in f([y_{i_k}, y']) \subseteq U(P, \varepsilon, A)$. This contradicts the definition of $\{y_i\}$. So for any $k \geq 2$, one has $y' \notin Y_{i_k}$. This implies that $[y_{i_1}, y_{i_k}] \subseteq Y_{i_k}$. By Lemma 3.11, there are $x \in (y', y_{i_1})$, a sufficiently big k and a (P, ε)-star Y' such that

$$[y_{i_k}, y_{i_1}] \supseteq [x, y_{i_1}] \xrightarrow{(f,\varepsilon)} Y'.$$

Hence $[y_{i_k}, y_{i_1}] \in \mathbb{Y}(\varepsilon, A)$. By Lemma 3.8-(3), one has $f([y_{i_k}, y_{i_1}]) \in \mathbb{Y}(\varepsilon, A)$ and $f(y_{i_k}) \in f([y_{i_k}, y_{i_1}]) \subseteq U(P, \varepsilon, A)$. This also contradicts the definition of $\{y_i\}$. Thus the proof is completed. □

Now it is time to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. As discussed at the beginning of this section, one can assume Condition I-III hold. Choose positive numbers $\delta_1 \geq \delta_2 \geq \ldots$ such that $\delta_1 < \varepsilon_0/2$, $f(B(x, \delta_1)) \subseteq B(f(x), \varepsilon_0)$ and $\lim_{n\to\infty} \delta_n = 0$. And choose points $w_1, w_2, \ldots$ in W such that for any $n \in \mathbb{N}$ one has $\overline{\{w_n, w_{n+1}, \ldots\}} = W$. For any $n \in \mathbb{N}$, by Lemma 3.12 and Lemma 3.7-(3), there are (P, δ_n)-arc A_n, $Y_n \in \mathbb{Y}(\delta_n)$ and a (P, δ_n)-star Y_n' such that

$$A_n \xrightarrow{(f,\delta_n)} Y_n \xrightarrow{(f,\delta_n)} Y_n' \supseteq A_{n+1} \text{ and } d(w_n, Y_n) < \delta_n/2.$$

It is easy to check $Y_1, Y_2, \ldots$ satisfy the condition of Lemma 3.5. Hence by Lemma 3.5, W is an ω-limit set. So the proof is completed. □

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