PRODUCT RECURRENT PROPERTIES, DISJOINTNESS AND WEAK DISJOINTNESS

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Abstract. Let $\mathcal{F}$ be a collection of subsets of $\mathbb{Z}_+$ and (X,T) be a dynamical system. $x \in X$ is $\mathcal{F}$-recurrent if for each neighborhood U of x, $\{n \in \mathbb{Z}_+ : T^n x \in U\} \in \mathcal{F}$. x is $\mathcal{F}$-product recurrent if (x,y) is recurrent for any $\mathcal{F}$-recurrent point y in any dynamical system (Y,S). It is known that x is $\{infinite\}$-product recurrent if and only if it is minimal and distal. In this paper it is proved that the closure of a $\{syndetic\}$-product recurrent point (i.e. weakly product recurrent point) has a dense minimal points; and a $\{piecewise\ syndetic\}$-product recurrent point is minimal. Moreover, it is observed that if (X,T) is disjoint from all minimal systems, then each transitive point of (X,T) is weakly product recurrent.

Results on product recurrence when the closure of an $\mathcal{F}$-recurrent point has zero entropy are obtained. Several results on disjointness are given, and results on weak disjointness are described when considering disjointness.

Contents

1. Introduction

1.1. **Dynamical preliminaries.** In the article, integers, nonnegative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ respectively. By a *topological dynamical*

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system (t.d.s.) we mean a pair (X,T), where X is a compact metric space (with metric d) and $T : X \to X$ is continuous and surjective. A non-vacuous closed invariant subset $Y \subset X$ defines naturally a *subsystem* (Y,T) of (X,T).

The *orbit* of x, $orb(x,T)$, is the set $\{T^n x : n \in \mathbb{Z}_+\} = \{x, T(x), \dots\}$. The *$\omega$-limit set* of x, $\omega(x,T)$, is the set of all limit points of $orb(x,T)$. It is easy to verify that $\omega(x,T) = \bigcap_{n\geq 0} \overline{\{T^i(x) : i \geq n\}}$.

A t.d.s. (X,T) is *transitive* if for each pair of opene (i.e. nonempty and open) subsets U and V, $N(U,V) = \{n \in \mathbb{Z}_+ : T^{-n}V \cap U \neq \emptyset\}$ is infinite. It is *point transitive* if there exists $x \in X$ such that $\overline{orb(x,T)} = X$; such x is called a *transitive point*. It is well known if (X,T) is transitive then the set of transitive points is a dense G_δ set (denoted by $Tran_T$). (X,T) is *weakly mixing* if $(X \times X, T \times T)$ is transitive.

A t.d.s (X,T) is *minimal* if $Tran_T = X$. Equivalently, (X,T) is minimal if and only if it contains no proper subsystems. By the well-known Zorn's Lemma argument any t.d.s (X,T) contains some minimal subsystem, which is called a *minimal set* of X. A point $x \in X$ is *minimal* or *almost periodic* if the subsystem $(\overline{orb(x,T)}, T)$ is minimal.

Let (X,T) be a t.d.s. Fix $(x,y) \in X^2$. It is a *proximal* pair if there is a sequence $\{n_i\}$ in $\mathbb{Z}_+$ such that $\lim_{n\to+\infty} T^{n_i}x = \lim_{n\to+\infty} T^{n_i}y$; it is a *distal* pair if it is not proximal. Denote by $P(X,T)$ the set of all proximal pairs of (X,T). A point x is said to be *distal* if whenever y is in the orbit closure of x and (x,y) is proximal, then $x = y$. A t.d.s. (X,T) is called *distal* if (x,x') is distal whenever $x, x' \in X$ are distinct.

A t.d.s (X,T) is *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_1,x_2) < \delta$ implies $d(T^n x_1, T^n x_2) < \epsilon$ for every $n \in \mathbb{Z}_+$. It is easy to see that each equicontinuous system is distal.

For a t.d.s. (X,T), $x \in X$ and $U \subset X$ let

$$N(x,U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}.$$

A point $x \in X$ is said to be *recurrent* if for every neighborhood U of x, we have $N(x,U)$ is infinite. Equivalently, $x \in X$ is recurrent if and only if $x \in \omega(x,T)$, i.e. there is a strictly increasing subsequence $\{n_i\}$ of $\mathbb{N}$ such that $T^{n_i}x \longrightarrow x$. Denote by $R(X,T)$ the set of all recurrent points of (X,T).

1.2. **Product recurrence and weakly product recurrence.** The notion of product recurrence was introduced by Furstenberg in [15]. Let (X,T) be a t.d.s. A point $x \in X$ is said to be *product recurrent* if given any recurrent point y in any dynamical system (Y,S), (x,y) is recurrent in product system $(X \times Y, T \times S)$. By associating product recurrence with a combinatorial property of sets of return times (i.e. x is product recurrent if and only if it is IP^* recurrent), Furstenberg [15] proved that product recurrence is equivalent to distality. In [6] Auslander and Furstenberg extended the equivalence of product recurrence and distality to more general semigroup actions.

If a semigroup E acts on the space X and F is a closed subsemigroup of E , then $x \in X$ is said to be *F-recurrent* if $px = x$ for some $p \in F$, and *product F-recurrent* if whenever y is an F-recurrent point (in some space Y on which E acts) the point (x, y) in the product system is F-recurrent. In [6] it is shown that, under certain conditions, a point is product F-recurrent if and only if it is a distal point. This is also discussed in [12].

In [6], Auslander and Furstenberg posed a question: if (x, y) is recurrent for all minimal points y, is x necessarily a distal point? This question is answered in the negative in [19]. Such x is called a *weakly product recurrent* point there.

The main purpose of this paper is to study more general case, i.e. to study a point x with property that (x, y) is recurrent for any y with some special recurrent property. We will also show how this approach is related to disjointness and weak disjointness. To be more precise, we need some notions.

1.3. **Furstenberg families.** Let us recall some notions related to a family (for details see [1, 15]). Let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upwards. That is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ can generate a family $[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$. If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a *filter*. For a family $\mathcal{F}$, the *dual family* is

$$\mathcal{F}^* = \{F \in \mathcal{P} : \mathbb{Z}_+ \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \ for\ all\ F' \in \mathcal{F}\}.$$

$\mathcal{F}^*$ is a family, proper if $\mathcal{F}$ is. Clearly,

$$(\mathcal{F}^*)^* = \mathcal{F} \text{ and } \mathcal{F}_1 \subset \mathcal{F}_2 \Longrightarrow \mathcal{F}_2^* \subset \mathcal{F}_1^*.$$

The family consisting of all infinite subsets of $\mathbb{Z}_+$ is denoted by $\mathcal{F}_{inf}$.

1.4. **$\mathcal{F}$-recurrence and some important families.** Let $\mathcal{F}$ be a family and (X, T) be a t.d.s. We say $x \in X$ is *$\mathcal{F}$-recurrent* if for each neighborhood U of x, $N(x, U) \in \mathcal{F}$. So the usual recurrent point is just $\mathcal{F}_{inf}$-recurrent one.

Recall that a t.d.s. (X, T) is

- an *E-system* if it is transitive and has an invariant measure μ with full support, i.e., $supp(\mu) = X$;
- an *M-system* if it is transitive and the set of minimal points is dense; and
- a P-system if it is transitive and the set of periodic points is dense.

A subset S of $\mathbb{Z}_+$ is *syndetic* if it has a bounded gaps, i.e. there is N such that $\{i, i+1, \cdots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}_+$. S is *thick* if it contains arbitrarily long runs of positive integers, i.e. there is a strictly increasing subsequence $\{n_i\}$ of $\mathbb{N}$ such that $S \supset \bigcup_{i=1}^{\infty}\{n_i, n_i+1, \ldots, n_i+i\}$. The collection of all syndetic (resp. thick) subsets is denoted by $\mathcal{F}_s$ (resp. $\mathcal{F}_t$). Note that $\mathcal{F}_s^* = \mathcal{F}_t$ and $\mathcal{F}_t^* = \mathcal{F}_s$. Some dynamical properties can be interrupted by using the notions of syndetic or thick subsets. For example, a classic result of Gottschalk stated that x is a minimal point if and only if $N(x,U) \in \mathcal{F}_s$ for any neighborhood U of x, and a t.d.s. (X,T) is weakly mixing if and only if $N(U,V) \in \mathcal{F}_t$ for any non-empty open subsets of X, [15].

A subset S of $\mathbb{Z}_+$ is *piecewise syndetic* if it is an intersection of a syndetic set with a thick set. Denote the set of all piecewise syndetic sets by $\mathcal{F}_{ps}$. It is known that a t.d.s. (X,T) is an *M-system* if and only if there is a transitive point x such that if U is a neighborhood of x then $N(x,U) \in \mathcal{F}_{ps}$, see [24, Lemma 2.1].

Let $\{b_i\}_{i\in I}$ be a finite or infinite sequence in $\mathbb{N}$. One defines

$$FS(\{b_i\}_{i\in I}) = \Big\{ \sum_{i\in\alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \Big\}.$$

F is an *IP set* if it contains some $FS(\{p_i\}_{i=1}^{\infty})$, where $p_i \in \mathbb{N}$. The collection of all IP sets is denoted by $\mathcal{F}_{ip}$. A subset of $\mathbb{N}$ is called an *IP*-set*, if it has non-empty intersection with any IP-set. It is known that a point x is a recurrent point if and only if $N(x,U) \in \mathcal{F}_{ip}$ for any neighborhood U of x, and x is distal if and only if x is IP^*-recurrent [15].

Let S be a subset of $\mathbb{Z}_+$. The *upper Banach density of* S is

$$BD^*(S) = \limsup_{|I|\to+\infty} \frac{|S\cap I|}{|I|},$$

where I ranges over intervals of $\mathbb{Z}_+$, while the *upper density* of S is

$$D^*(S) = \limsup_{n\longrightarrow\infty} \frac{|S\cap[0,n-1]|}{n}.$$

Let $\mathcal{F}_{pubd} = \{S \subseteq \mathbb{Z}_+ : BD^*(S) > 0\}$. It is known a t.d.s. is an *E-system* if and only if there is a transitive point x such that if U is a neighborhood of x then $N(x,U) \in \mathcal{F}_{pubd}$, see [22, Lemma 3.6].

1.5. **$\mathcal{F}$-product recurrence and disjointness.** Let $\mathcal{F}$ be a family. For a t.d.s. (X,T), $x \in X$ is *$\mathcal{F}$-product recurrent* if given any $\mathcal{F}$-recurrent point y in any t.d.s (Y,S), (x,y) is recurrent in product system $(X\times Y, T\times S)$. Note that $\mathcal{F}_{inf}$-product recurrence is nothing but product recurrence; and $\mathcal{F}_s$-product recurrence is weakly product recurrence. In this paper we will study the properties of $\mathcal{F}$-product recurrent points, especially when $\mathcal{F} = \mathcal{F}_{pubd}$, $\mathcal{F}_{ps}$, or $\mathcal{F}_s$.

The notion of *disjointness* of two t.d.s. was introduced by Furstenberg [14]. If (X,T) and (Y,S) are two t.d.s. we say $J \subset X\times Y$ is a *joining* of X and Y if J is a non-empty closed invariant set and is projected onto X and Y. If each joining is equal to $X\times Y$ then we say that (X,T) and (Y,S) are *disjoint*, denoted by $(X,T) \perp (Y,S)$ or $X \perp Y$. It is known that if $(X,T)\perp(Y,S)$ then one of them is minimal [14], and if (X,T) is minimal then the set of recurrent points of (Y,S) is dense [24].

It turns out that if a transitive t.d.s. (X,T) is disjoint from all minimal t.d.s. then each transitive point of (X,T) is a weak product recurrent one. Thus, by [24] it is not necessarily minimal. Moreover, it is proved that the orbit closure of each weak product recurrent point is an M-system, i.e. with a dense set of minimal points. Contrary to the above situation it is shown that an $\mathcal{F}_{ps}$-product recurrent point is minimal.

Results on product recurrence when the closure of an $\mathcal{F}$-recurrent point has zero entropy are obtained. We will show that if (x, y) is recurrent for any point y whose orbit closure is a minimal system having zero entropy, then x is $\mathcal{F}_{pubd}$-recurrent, and if (x, y) is recurrent for any point y whose orbit closure is an M-system having zero entropy, then x is minimal. Moreover, it turns out that if (x, y) is recurrent for any recurrent y whose orbit closure has zero entropy, then x is distal.

Also several results on disjointness are given, and results on weak disjointness are described when considering disjointness. For example, we will show that a weakly mixing system with dense minimal points is disjoint from all minimal PI systems; a weakly mixing system with a dense set of distal points or an $\mathcal{F}_s$-independent t.d.s. is disjoint from any minimal t.d.s.; and if a transitive t.d.s. is disjoint from all minimal weakly mixing t.d.s. then it is an M-system.

1.6. **Organization of the paper.** The paper is organized as follows: In Section 2. we discuss recurrence and product recurrence. We begin with Hindman Theorem and rebuilt Furstenberg's result about product recurrence. In Section 3. we study $\mathcal{F}_{ps}$-product recurrence and show any $\mathcal{F}_{ps}$-product recurrent point is minimal. In Section 4. we aim to show that the closure of an $\mathcal{F}_s$-product recurrent point is an M-system. On the way to do this, we show that if (X, T) is a transitive t.d.s. which is disjoint from any minimal system, then each point in $Tran_T$ is $\mathcal{F}_s$-product recurrent. Thus combining results from [24] we reprove that an $\mathcal{F}_s$-product recurrent point is not necessarily minimal which obtained in [19]. In Section 5 $\mathcal{F}$-product recurrence with zero entropy is discussed. Some properties concerning extensions and factors are discussed in Section 6. We study disjointness and weak disjointness in Section 7. In Section 8 we discuss some more generalizations of the notions concerning product recurrence. Finally in the Appendix the relative proximal cell is discussed.

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2. Recurrence and product recurrence

It is known that x is distal if and only if (x, y) is recurrent for any recurrent point y. The usual proof uses the Auslander-Ellis theorem which states that if (X, T) is a t.d.s. and $x \in X$ then there is a minimal point $y \in X$ such that (x, y) is proximal. Usually one proves the Auslander-Ellis theorem by using the Ellis semigroup theory. In this section we give a proof of the theorem without using the Ellis semigroup theory.

2.1. **Recurrence and IP-set.** In this subsection we will use Hindman Theorem to prove Auslander-Ellis Theorem. Also some relations between recurrence and IP-set will be built.

Theorem 2.1 (Hindman)**.** [20] *For any finite partition of an IP-set, one of the cells of the partition still contains an IP-set.*

The following lemma is basically due to Furstenberg, see [15].

Lemma 2.2. *Let (X,T) be a compact metric t.d.s. If $x \in R(X,T)$ and $\{V_i\}_{i=1}^{\infty}$ is a collection of neighborhood of x, then there is some IP set $FS(\{p_i\}_{i=1}^{\infty})$ such that $FS(\{p_i\}_{i=n}^{\infty}) \subset N(x,V_n)$ for all $n \in \mathbb{N}$. Especially, each recurrent point is $\mathcal{F}_{ip}$-recurrent.*

Proof. We prove inductively. Since V_1 is a neighborhood of x and x is recurrent, there is some $p_1 \in \mathbb{N}$ such that

$$T^{p_1}x \in V_1.$$

As $V_1, T^{-p_1}V_1, V_2$ are neighborhood of x, so is their intersection $V_1 \cap T^{-p_1}V_1 \cap V_2$. And by recurrence of x there is some $p_2 \in \mathbb{N}$ such that

$$T^{p_2}x \in V_1 \cap T^{-p_1}V_1 \cap V_2.$$

Hence

$$T^{p_1}x, T^{p_2}x, T^{p_1+p_2}x \in V_1,$$

and

$$T^{p_2}x \in V_2.$$

Now for $n \in \mathbb{N}$ assume that we have sequence $p_1, p_2, \ldots, p_n$ such that

$$FS(\{p_i\}_{i=j}^{n}) \subseteq N(x,V_j), j = 1,2,\ldots,n. \tag{2.1}$$

That is, for each $j = 1,2,\ldots,n$

$$T^m x \in V_j, \quad \forall m \in FS(\{p_i\}_{i=j}^{n}).$$

Hence $\left(\bigcap_{j=1}^{n} \bigcap_{m \in FS(\{p_i\}_{i=j}^{n})} T^{-m}V_j\right) \cap \bigcap_{i=1}^{n+1} V_i$ is a neighborhood of x. Take $p_{n+1} \in \mathbb{N}$ such that

$$T^{p_{n+1}}x \in \left(\bigcap_{j=1}^{n} \bigcap_{m \in FS(\{p_i\}_{i=j}^{n})} T^{-m}V_j\right) \cap \bigcap_{i=1}^{n+1} V_i.$$

Then for each $j = 1,2,\ldots,n+1$

$$T^m x \in V_j, \quad \forall m \in FS(\{p_i\}_{i=j}^{n+1}).$$

That is

$$FS(\{p_i\}_{i=j}^{n+1}) \subseteq N(x,V_j), j = 1,2,\ldots,n+1.$$

So inductively we have an IP set $FS(\{p_i\}_{i=1}^{\infty})$ such that $FS(\{p_i\}_{i=n}^{\infty}) \subset N(x,V_n)$ for all $n \in \mathbb{N}$. And the proof is completed. □

Let (X,T) be a t.d.s. and $A \subseteq \mathbb{Z}_+$ be a sequence. Write

$$T^A x = \{T^n x : n \in A\}$$

and let $A - n = \{m - n : m \in A, m - n \geq 1\}$ for $n \in \mathbb{Z}_+$.

Using the method from [11], we have

Lemma 2.3. *Let (X,T) be a compact metric t.d.s. and $P = FS(\{p_i\}_{i=1}^\infty)$. For any $x \in X$ there is some $y \in \overline{T^P x} \cap R(X,T)$ and $\{p_{n_i}\}_{i=1}^\infty \subseteq \{p_i\}_{i=1}^\infty$ such that for any neighborhood U of y there is some j with $FS(\{p_{n_i}\}_{i=j}^\infty) \subseteq N(y,U)$ and $(x,y) \in P(X,T)$.*

Proof. Set $K_1 = \overline{T^P x}, P_1 = P$ and $p_{n_i} \in \{p_i\}_{i=1}^\infty$. Then

$$P_1 \cap (P_1 - p_{n_1}) \supseteq FS(\{p_i\}_{i \neq n_1}).$$

Hence

$$K_1 \cap T^{-p_{n_1}} K_1 \supseteq \overline{T^{P_1 \cap (P_1 - p_{n_1})} x}.$$

Let $K_1 \cap T^{-p_{n_1}} K_1 = \bigcup_{i=1}^{r_1} K_{1,i}$, where $K_{1,i}$ is compact and $diam K_{1,i} < \frac{1}{2}$. So we have

$$P_1 \cap (P_1 - p_{n_1}) = \bigcup_{i=1}^{r_1} \{n \in P_1 \cap (P_1 - p_{n_1}) : T^n x \in K_{1,i}\}.$$

By Hindman Theorem there is some j such that $P_2 = \{n \in P_1 \cap (P_1 - p_{n_1}) : T^n x \in K_{1,j}\}$ is an IP subset of $P_1 \cap (P_1 - p_{n_1})$. And we set $K_2 = K_{1,j}$. Clearly, $K_2 \subseteq K_1$, $diam K_2 < \frac{1}{2}$, $T^{p_{n_1}} K_2 \subseteq K_1$ and $T^{P_2} x \subseteq K_2$.

Continuing inductively, we have $\{p_{n_i}\} \subseteq \{p_i\}$, IP sets $P_1 \supseteq P_2 \supseteq \cdots$ and compact sets $K_1 \supseteq K_2 \supseteq \cdots$ such that $diam K_j < \frac{1}{j}$, $p_{n_j} \in P_j$, $T^{p_{n_j}} K_{j+1} \subseteq K_j$ and $T^{P_j} x \subseteq K_j$. Let $y \in \bigcap_{i=1}^\infty K_i$. It is easy to check it satisfies require. □

Proposition 2.4. *Let (X,T) be a compact metric t.d.s.. If (Y,S) is another t.d.s. and $z \in R(Y,S)$, then for any $x \in X$ there is some $y \in \overline{orb(x)}$ such that $(x,y) \in P(X,T)$ and (y,z) is a recurrent point of $X \times Y$.*

Proof. Let $\{V_n\}_{n=1}^\infty$ be neighborhood basis of z and by Lemma 2.2 there is some IP set $P = FS(\{p_i\}_{i=1}^\infty)$ such that $FS(\{p_i\}_{i=n}^\infty) \subset N(z, V_n)$ for all $n \in \mathbb{N}$. Let y be the recurrent point described in Lemma 2.3. Then for any neighborhoods U, V of y, z we have

$$N((y,z), U \times V) = N(y,U) \cap N(z,V) \neq \emptyset.$$

Hence (y,z) is a recurrent point of $X \times Y$. □

Theorem 2.5 (Auslander-Ellis)**.** *Let (X,T) be a compact metric t.d.s.. Then for any $x \in X$ there is some minimal point $y \in \overline{orb(x)}$ such that (x,y) is proximal.*

Proof. Without loss of generality, we assume x is not minimal. First we claim there is some minimal set Y in $\overline{orb(x)}$. Now we will find a thick A such that $\overline{T^A x} \setminus T^A x \subseteq Y$. Then taking any IP subset P from A, by Lemma 2.3 there is some $y \in \overline{T^P x} \cap R(X)$

and $(x, y) \in P_X$. Since $y \in \overline{T^P x} \setminus T^P x \subseteq Y$, y is a minimal point. Thus we finish our proof.

Let $V_n = \{z \in X : d(z, Y) < \frac{1}{n}\}$ and then $\{V_n\}_{n=1}^{\infty}$ is the neighborhood basis of Y. Let $\delta_n > 0$ such that $d(T^i x', T^i x'') < \frac{1}{n}, i = 0, 1, \cdots, n-1$ if only $d(x', x'') < \delta_n$. As $Y \subseteq \overline{orb(x)}$ there is some i_n such that $d(T^{i_n} x, Y) < \delta_n$. Then by Y is invariant $d(T^{i_n+j} x, Y) < \frac{1}{n}, j = 0, 1, \cdots, n-1$. Set $A = \bigcup_{n=1}^{\infty} \{i_n + j\}_{j=0}^{n-1}$. By our construction we have $\overline{T^A x} \setminus T^A x \subseteq Y$. □

Remark 2.6. (1) The previous proofs of Theorem2.5 use Zorn's Lemma more or less. Here for compact metric space we get the proof only using Hindman Theorem. Note that usually to show that any t.d.s (X, T) contains some minimal subsystem is using the well-known Zorn's Lemma argument. But for the case when X is metric and action semigroup is $\mathbb{Z}_+$ Weiss [34] gave a constructive proof.

(2) From Auslander-Ellis Theorem Furstenberg introduce a so-called central set. A subset $S \subseteq \mathbb{Z}_+$ is a *central set* if there exists a system (X, T), a point $x \in X$ and a minimal point y proximal to x, and a neighborhood U_y of y such that $N(x, U_y) \subset S$. It is known that any central set contains an IP-set[15].

(3) By Lemma 2.2 x is a recurrent point if and only if it is $\mathcal{F}_{ip}$-recurrent. In [15, Theorem 2.17] it is also shown that for any IP-set R there exist a t.d.s (X, T), a recurrent $x \in X$ and a neighborhood U of x such that $N(x, U) \subseteq R \cup \{0\}$.

2.2. **Product recurrence.** The following proposition was proved in [15] and we give a proof for completeness.

Proposition 2.7. *The following statements are equivalent:*

(1) *x is distal.*
(2) *x is product recurrent.*
(3) *(x, y) is minimal for each minimal point y.*
(4) *x is IP^*-recurrent.*

Proof. Denote $X = \overline{orb(x)}$. First by Remark 2.6 it is easy to see that (2)$\Longleftrightarrow$(4).

(1) $\Longrightarrow$ (4). If x is not IP^*-recurrent, then there is a neighborhood U of x such that $N(x, U)$ is not IP^*-set, i.e. there exists an IP-set P such that $T^P x \cap U = \emptyset$. By Lemma2.3, we know that there is a point $y \in \overline{T^P x}$ i.e. $y \notin U$ such that $(x, y) \in P(X, T)$ which contradicts that x is distal.

(4) $\Longrightarrow$ (1). As any thick set contains an IP-set, we get that x is a minimal point. If x is not distal, there exists a different point $x' \in X$ such that $(x, x') \in P(X, T)$. Let U and U' be any neighborhood of x and x' which disjoint. $N(x, U')$ is a central set and contains an IP-set, so $N(x, U) \cap N(x, U') \neq \emptyset$ which implies $x = x'$.

(1) $\Longrightarrow$ (3). Let y be an any minimal point. If (x, y) is not minimal, by Lemma**??** there exists a minimal point $(x', y') \in \overline{orb((x, y))}$ which is proximal to (x, y). It is easy to show that x' is proximal to x which implies $x = x'$. For any neighborhood $U \times V$ of (x, y), $N(x, U)$ is IP^*-set and $N(y', V)$ is a central set as y' is proximal to

minimal point y, so we know that $N(x,U)\cap N(y',V)\neq\emptyset$ i.e. $(x,y)\in\overline{orb((x,y'))}$ which implies that (x,y) is a minimal point.

(3) $\Longrightarrow$ (1). It is easy to get that x is a minimal point. If there exists a point $x'\in\overline{orb(x)}$ which is proximal to x, so there is a point $(y,y)\in\overline{orb((x,x'))}$. As (x,x') is a minimal point, then $(x,x')\in\overline{orb((y,y))}$ which implies $x=x'$, so x is distal. □

3. $\mathcal{F}_{ps}$-PRODUCT RECURRENT POINTS

In this section we aim to show that if x is a $\mathcal{F}_{ps}$-product recurrent point then it is minimal.

Definition 3.1. Let (X,T) be a t.d.s. and $\mathcal{F}$ be a family. $x\in X$ is *$\mathcal{F}$-product recurrent* ($\mathcal{F}$-PR for short) if given any $\mathcal{F}$-recurrent point y in any t.d.s (Y,S), (x,y) is recurrent in product system $(X\times Y, T\times S)$.

By definition we have the following immediately.

Lemma 3.2. *Let $\mathcal{F}_1,\mathcal{F}_2$ be two families with $\mathcal{F}_1\subseteq\mathcal{F}_2$. Then each $\mathcal{F}_2$-PR point is $\mathcal{F}_1$-PR.*

It is clear that

$$\mathcal{F}_{inf}-PR \;\Rightarrow\; \mathcal{F}_{pubd}-PR \;\Rightarrow\; \mathcal{F}_{ps}-PR \;\Rightarrow\; \mathcal{F}_{s}-PR.$$

It was shown in [19] that an $\mathcal{F}_s$-PR point is not necessarily minimal (more examples will be given in the next section). A natural question is: if x is $\mathcal{F}_{ps}$-PR, is x minimal? To settle down this question we need the following notions. By an *md-set* A we mean there is an M-system (Y,S), a transitive point $y\in Y$ and a neighborhood U of y with $A=N(y,U)$.

Before continuing discussion, we need some preparation about symbolic dynamics. Let $\Sigma_2=\{0,1\}^{\mathbb{Z}_+}$ and $\sigma:\Sigma_2\longrightarrow\Sigma_2$ be the shift map, i.e. the map

$$(x_0,x_1,x_2,x_3,\ldots)\mapsto(x_1,x_2,x_3,\ldots)\in\Sigma_2.$$

A *shift space* (X,σ) is a subsystem of (Σ_2,σ). And if (X,σ) is a shift space, let $[i]=\{x\in X: x(0)=i\}$ for $i=0,1$ and $[A]=\{x\in X: x_0x_1\cdots x_{(l-1)}=A\}$ for any finite block A where l is the length of A. For any $S\subset\mathbb{Z}_+$, we denote 1_S be the indicator function from $\mathbb{Z}_+$ to $\{0,1\}$, i.e. $1_S(s)=1$ if $s\in S$ and $1_S(s)=0$ if $s\notin S$. In a natural way, each indicator function can be regarded as an element of $\{0,1\}^{\mathbb{Z}_+}$. For finite blocks $A=(a_1,\ldots,a_n)\in\{0,1\}^n$ and $B=(b_1,\ldots,b_n)\in\{0,1\}^n$ we say $A\leq B$ if $a_i\leq b_i$ for each $i\in\{1,2,\cdots,n\}$. For finite blocks A and B we denote the length of A by $|A|$ and denote $A^n=\underbrace{AA\cdots A}_{n}$ for $n\in\mathbb{N}$ where AB denotes the concatenation of A and B. In particular $0^n=\underbrace{00\cdots 0}_{n}$.

Proposition 3.3. *Every thick set containing* 0 *contains an md-set.*

Proof. Let $C \subset \mathbb{Z}_+$ be a thick set with $0 \in C$. Let $x = 1_C = (x_0, x_1, \dots) \in \{0,1\}^{\mathbb{Z}_+}$.

By the assumption $x_0 = 1$ and there are $p_n, q_n \in \mathbb{N}$ with $(1, 1, \dots, 1) \le (x_{p_n}, \dots, x_{q_n})$ for any $n \in \mathbb{N}$. It is clear that there is $a_1 \ge 1$ such that

$$A_1 = 10^{a_1}1 \le (x_0 \dots x_{l_1})$$

with $l_1 = |A_1| - 1$. By the same reasoning there is $a_2 > a_1$ and a_2 can be divided by $|A_1|$ with

$$A_2 = A_1 0^{a_2} A_1 \le (x_0, \dots, x_{l_2})$$

where $l_2 = |A_2| - 1$. Then $|A_2|$ can be divided by $|A_1|$.

Inductively assume that $A_1, \dots, A_k$ are defined, then there is $a_{k+1} > a_k$ and a_{k+1} can be divided by $|A_k|$ with

$$A_{k+1} = A_k 0^{a_k} A_k A_{k-1}^{n_{k+1}^{k-1}} \dots A_2^{n_{k+1}^2} A_1^{n_{k+1}^1} \le (x_0, \dots, x_{l_{k+1}})$$

where $|A_1|^{n_{k+1}^1} = |A_2|^{n_{k+1}^2} = \dots = |A_{k-1}|^{n_{k+1}^{k-1}} = |A_k|$ and $l_{k+1} = |A_{k+1}| - 1$. Then $|A_{k+1}|$ can be divided by $|A_j|$ for $1 \le j \le k$. It is easy to see that $\forall i \in \mathbb{N}, n_j^i \to \infty$ when $j \to \infty$.

Let $y = \lim_{k\to\infty} A_k \in \{0,1\}^{\mathbb{Z}_+}$, then y is a recurrent point under the shift σ. It is clear that $N(y, [A_n])$ is piece-wise syndetic. Thus the orbit closure of y is an M-system (in fact it is a P-system). At the same time,

$$N(y, [1]) = \{n \in \mathbb{Z}_+ : \sigma^n y \in [1]\} \subset C.$$

This completes the proof. □

Now we give a positive answer to the question.

Theorem 3.4. *Let (X,T) be a t.d.s.. If x is $\mathcal{F}_{ps}$-PR, then it is minimal.*

Proof. If x is not minimal, then there is a neighborhood U of x such that $N(x,U)$ is not syndetic. Thus, $\mathbb{Z}_+ \setminus N(x,U)$ is thick. Let $C = \{0\} \cup \mathbb{Z}_+ \setminus N(x,U)$. By Proposition 3.3, C contains a subset $A = N(y,[1])$, where y is the point defined in Proposition 3.3 which is $\mathcal{F}_{ps}$-recurrent. Then

$$N((x,y), U \times [1]) = N(x,U) \cap N(y,[1]) \subset \{0\},$$

which implies that (x,y) is not recurrent, a contradiction. Thus x is minimal. □

Since each $\mathcal{F}_{pubd}$-PR point is an $\mathcal{F}_{ps}$-PR one, as a corollary of Theorem 3.4, each $\mathcal{F}_{pubd}$-PR point is minimal. Generally, we have

Corollary 3.5. *Let $\mathcal{F}$ be a family with $\mathcal{F}_{ps} \subseteq \mathcal{F}$. Then each $\mathcal{F}$-PR point is minimal.*

4. $\mathcal{F}_s$-product recurrent points

In this section we aim to show that the closure of an $\mathcal{F}_s$-product recurrent point is an M-system. On the way to do this, we show that if (X,T) is a transitive t.d.s. which is disjoint from any minimal system, then each point in $Tran_T$ is $\mathcal{F}_s$-PR. Thus combining results from [24] we reprove that an $\mathcal{F}_s$-PR point is not necessarily minimal which was obtained in [19]. Note that weak product recurrence is also discussed in [30] recently.

4.1. $\mathcal{F}_s$-product recurrence.

Definition 4.1. A subset A of $\mathbb{Z}_+$ is called an *m-set*, if there exist a minimal system (Y,S), $y \in Y$ and a non-empty open subset V of Y such that $A \supset N(y,V)$. The family generated by all m-sets is denoted by $\mathcal{F}_{mset}$.

A subset A of $\mathbb{Z}_+$ is called an *sm-set* (standing for standard m-set), if there exist a minimal system (Y,S), $y \in Y$ and an open neighborhood V of y such that $A \supset N(y,V)$. The family generated by all sm-set is denoted by $\mathcal{F}_{smset}$.

It is clear that $\mathcal{F}_{smset} \subset \mathcal{F}_{mset}$ and hence $\mathcal{F}^*_{mset} \subset \mathcal{F}^*_{smset}$. We will show that $\mathcal{F}^*_{smset} \subset \mathcal{F}_{ip}$. Moreover, we have the following observation.

Proposition 4.2. *The following statements hold.*

(1) *Let (X,T) be transitive and $x \in Tran_T$. Then (X,T) is disjoint from any minimal t.d.s. if and only if $N(x,U) \cap A \neq \emptyset$ for any neighborhood U of x and any m-set A, i.e. $N(x,U) \in \mathcal{F}^*_{mset}$.*
(2) *a point x is $\mathcal{F}_s$-PR if and only if for each open neighborhood U of x and each sm-set A, $N(x,U) \cap A \neq \emptyset$, i.e. $N(x,U) \in \mathcal{F}^*_{smset}$.*

Proof. (1) is proved in [24]. (2) follows from the definitions. □

So we have

Theorem 4.3. *Let (X,T) be a transitive t.d.s. which is disjoint from any minimal system. Then each point in $Tran_T$ is $\mathcal{F}_s$-PR and non-minimal.*

Proof. It follows by Proposition 4.2 directly. We give a direct argument here. Let $x \in Tran_T$ and (Y,S) be a given minimal t.d.s.. For $y \in Y$ let $A = \overline{orb\big((x,y), T \times S\big)}$. It is clear that A is a joining and hence $A = X \times Y$. This implies that (x,y) a recurrent point of $(X \times Y, T \times S)$ and hence x is $\mathcal{F}_s$-PR. □

For a t.d.s. (X,T), $x \in X$ is a *regular minimal point* if for each neighborhood U of x, there is $k = k(U)$ such that $N(x,U) \supset k\mathbb{Z}_+$. In [24] Huang and Ye showed that any weakly mixing t.d.s. with a dense regular minima points is disjoint from any minimal t.d.s.. There are a lot of non-minimal systems with this properties, for example the full shift and the example constructed in [24]. Thus an $\mathcal{F}_s$-PR point is not necessarily minimal. We note that this result was also obtained in [19]. So naturally we would ask: if x is $\mathcal{F}_s$-PR and not minimal, what we can say about the properties of such point. In fact we will show that the closure of x is an M-system, i.e. it has a dense minimal points.

The way we answer the question is that we will show every thickly syndetic set containing $\{0\}$ contains an m-set. Note that a subset A of $\mathbb{Z}_+$ is *thickly syndetic* if it has non-empty intersection with any piece-wise syndetic set. More precisely, a subset of $\mathbb{Z}_+$ is thickly syndetic if for each $n \in \mathbb{N}$ there is a syndetic subset $S_n = \{s_1^n, s_2^n, \ldots\}$ such that $S \supset \cup_{n=1}^\infty \cup_{i=1}^\infty \{s_i^n + 1, s_i^n + 2, \ldots, s_i^n + n\}$.

For a transitive system whether it is disjoint from all minimal systems can be checked through m-sets, for the details see [24]. Particularly the authors showed that every thickly syndetic set containing $\{0\}$ contains an m-set. To solve our question we need to show

Proposition 4.4. *Every thickly syndetic set containing* $\{0\}$ *contains an sm-set.*

Since the proof of Proposition 4.4 is a little long, we left it to next subsection. Now we have

Theorem 4.5. *The orbit closure of an* $\mathcal{F}_s$*-PR point is an* M*-system.*

Proof. Let x be an $\mathcal{F}_s$-PR point and U be an open neighborhood of x. If $N(x,U)$ is not piece-wise syndetic, then $A = \mathbb{Z}_+ \setminus N(x,U)$ is thickly syndetic. Then by Proposition 4.4, $A \cup \{0\}$ contains $N(y,V)$, where (Y,S) is a minimal set, $y \in Y$ and V is an open neighborhood of y. Thus we have $N((x,y), U\times V) = N(x,U) \cap N(y,V) \subset \{0\}$, a contradiction. □

Remark 4.6. Recall that two t.d.s. (X,T) and (Y,S) are *weakly disjoint* if $(X \times Y, T\times S)$ is transitive. A t.d.s. is *scattering* if it is weakly disjoint from all minimal t.d.s. [9]. We remark that a transitive point in a non-minimal scattering t.d.s. is not necessarily weakly product recurrent, since there is an almost equicontinuous scattering t.d.s. which is not an M-system, see [28, Theorem 4.6]

We also have the following remark.

Remark 4.7. It is easy to see that if x is weakly product recurrent and y is distal, then (x,y) is also weakly product recurrent. This implies that $\overline{orb(x,y)}$ is not necessarily weakly mixing. Thus, the collection of sm-set is strictly contained in the collection of m-set, since if (X,T) is disjoint from all minimal t.d.s. then (X,T) is weakly mixing, see [24].

4.2. **Proof of Proposition 4.4.** Let $F \subset \mathbb{Z}_+$ be a thickly syndetic subset. We will construct $y^n = 1_{F_n} \in \{0,1\}^{\mathbb{Z}_+}$ such that $F_n \subset F$ and $y = \lim y^n = 1_A$ is a minimal point. Let $Y = \overline{orb(y,\sigma)}$ and $[1] = \{x \in Y : x(0) = 1\}$. As $A \subset F$ and $A = N(y,[1])$ the theorem follows.

To obtain y^n we construct a finite word A_n such that y^n begins with A_n, A_n appears in y^n syndetically and A_{n+1} begins with A_n. The reason we can do this is that $1^n = (1,\ldots,1)$ (n times) appears in 1_F syndetically for each $n \in \mathbb{N}$. More precisely we do as follows.

Step 1: Construct A_1 and $F_1 \subset F$ such that A_1 appears in $y^1 = 1_{F_1}$ with gaps bounded by l_1 and y^1 begins with A_1.

Let $\min F = a_1 - 1$ and $A_1 = 1_F[0; a_1 - 1]$. Set $B_1 = A_1 A_1 0 A_1$ and $r_1 = b_1 = |B_1| = 3a_1 + 1$. As F is thickly syndetic, 1^{r_1} appears in F at a syndetic set $W_1 = \{w^1_1, w^1_2, \ldots\}$. Without loss of generality assume that $2r_1 \le w^1_{j+1} - w^1_j \le l_1$ and $2k_1 \le w^1_1 \le l_1$, where l_1 is some number in $\mathbb{N}$. Put $u^1_i = w^1_i, i \in \mathbb{N}$. Choose $y^1 \in \{0,1\}^{\mathbb{Z}_+}$ such that

- $y^1[0; a_1 - 1] = A_1$, $y^1[u^1_i; u^1_i + b_1 - 1] = B_1$ and
- $y^1(j) = 0$ if $j \in \mathbb{Z}_+ \setminus ([0; a_1 - 1] \cup \cup_{i=1}^{\infty} [u^1_i; u^1_i + b_1 - 1])$.

It is easy to see that B_1 as well as A_1 appears in y^1 with gaps bounded by l_1 and $F_1 \subset F$, where $1_{F_1} = y^1$.

Step 2: Construct A_2 and $F_2 \subset F$ such that

(1) A_2 has the form of $A_1V_1B_1$ and if $a_2 = |A_2|$ then $A_2 = y^1[0; a_2 - 1]$.
(2) $y^2[0; a_2 - 1] = A_2$ and A_1, A_2 appear in y^2 syndetically with gaps bounded by l_1 and l_2 respectively.
(3) $F_2 = \{i \in \mathbb{Z}_+ : y^2(i) = 1\} \subset F$.

Set $a_2 = u^1_1 + b_1$ and let $A_2 = y^1[0; a_2 - 1]$, $B_2 = A_2A_20A_2$, $b_2 = |B_2| = 3a_2 + 1$. Then A_2 has the form of $A_1V_1B_1$. Let $r_2 = 2l_1 + 2b_1 + b_2$. As F is thickly syndetic, 1^{r_2} appears in F at a syndetic set $W_2 = \{w^2_1, w^2_2, \ldots\}$. Without loss of generality assume that $2r_2 \le w^2_{j+1} - w^2_j \le l_2 - (l_1 + b_1)$ and $2a_2 \le w^2_1 \le l_2 - (l_1 + b_1)$, where l_2 is some number in $\mathbb{N}$.

To get y^2 we change y^1 at places $[w^2_i; w^2_i + r_2 - 1]$ for each $i \in \mathbb{N}$. It is enough to show the idea how we do at $[w^2_1; w^2_1 + r_2 - 1]$.

Let k, j satisfy that $u^1_{k-1} < w^2_1 \le u^1_k$ and $u^1_j + b_1 - 1 \le w^2_1 + r_2 - 1 < u^1_{j+1} + b_1 - 1$. Let l be the integer part of $(u^1_j - 1 - u^1_k - b_1 - b_2)/b_1$.

Put $u^2_1 = u^1_k + b_1$. Let $y^2[u^2_1; u^2_1 + b_2 - 1] = B_2$ and $y^2[u^2_1 + b_2 + pb_1; u^2_1 + b_2 + (p + 1)b_1 - 1] = B_1$ for $p = 0, 1, \ldots, l - 1$. That is, first we put B_2 at place u^2_1 and then we put as many as B_1 we can. We do the same at all places $[w^2_i; w^2_i + r_2 - 1]$, we get $u^2_i \in [w^2_i, w^2_i + r_2 - 1]$ with $y^2[u^2_i; u^2_i + b_2 - 1] = A_2$, $i = 1, 2, \ldots$.

In such a way we get y^2. It is easy to see that y^1 and y^2 differ possibly at $[w^2_i; w^2_i + r_2 - 1]$. Thus

$$F_2 = \{i \in \mathbb{Z}_+ : y^2(i) = 1\} \subset F_1 \bigcup \cup_{i=1}^{\infty} [w^2_i; w^2_i + r_2 - 1].$$

At the same time B_1, B_2 appear in y^2 syndetically with gaps bounded by l_1 and l_2 respectively by the construction and so are A_1, A_2.

Step 3: Construct A_{m+1} and $F_{m+1} \subset F$ inductively such that

(1) A_{m+1} has the form of $A_mV_mB_m$ and if $a_{m+1} = |A_{m+1}|$ then $A_{m+1} = y^m[0; a_{m+1} - 1]$.
(2) $y^{m+1}[0; a_{m+1} - 1] = A_{m+1}$ and A_i appear in y^{m+1} syndetically with gaps bounded by l_i for each $1 \le i \le m + 1$.
(3) $F_{m+1} = \{i \in \mathbb{Z}_+ : y^{m+1}(i) = 1\} \subset F$.

Set $a_{m+1} = u^m_1 + b_m$ and let $A_{m+1} = y^m[0; a_{m+1} - 1]$, $B_{m+1} = A_{m+1}A_{m+1}0A_{m+1}$, and $b_{m+1} = |B_{m+1}| = 3a_{m+1} + 1$. Then A_{m+1} has the form of $A_mV_mB_m$. Let $r_{m+1} = 2l_m + 2b_m + b_{m+1}$. As F is thickly syndetic, $1^{r_{m+1}}$ appears in F at a syndetic set $W_{m+1} = \{w^{m+1}_1, w^{m+1}_2, \ldots\}$. Without loss of generality assume that $2r_{m+1} \le w^{m+1}_{j+1} - w^{m+1}_j \le l_{m+1} - (l_m + b_m)$ and $2k_{m+1} \le w^{m+1}_1 \le l_{m+1} - (l_m + b_m)$, where l_{m+1} is some number in $\mathbb{N}$.

To get y^{m+1} we change y^m at places $[w^{m+1}_i; w^{m+1}_i + r_{m+1} - 1]$ for each $i \in \mathbb{N}$. It is enough to show the idea how we do at $[w^{m+1}_1; w^{m+1}_1 + r_{m+1} - 1]$.

Let k, j satisfy that $u^m_{k-1} < w^{m+1}_1 \le u^m_k$ and $u^m_j + b_m - 1 \le w^{m+1}_1 + r_{m+1} - 1 < u^m_{j+1} + b_m - 1$.

Put $u_1^{m+1} = u_k^m + b_m$. Let $y^{m+1}[u_1^{m+1}; u_1^{m+1} + b_{m+1} - 1] = B_{m+1}$ and

$$y^{m+1}[u_1^{m+1}, u_j^m - 1] = B_{m+1}(B_m)^{p_m}(B_{m-1})^{p_{m-1}} \dots (B_1)^{p_1} C_{m+1},$$

where C_{m+1} is a word, and $p_1, \dots, p_m$ are natural numbers with

- $|C_{m+1}| < b_1$,
- $|C_{m+1}| + b_1 p_1 < b_2$, and
- $|C_{m+1}| + b_1 p_1 + \dots + b_i p_i < b_{i+1}$ for each $1 \le i \le m-1$.

That is, first we put B_{m+1} at place u_1^{m+1} and start from $u_1^{m+1} + k_{m+1}$ to u_j^m we put as many as B_m we can and then we put as many as B_{m-1} we can and so on. We do the same at all places $[w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$, we get $u_i^{m+1} \in [w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$ with $y^{m+1}[u_i^{m+1}; u_i^{m+1} + b_{m+1} - 1] = B_{m+1}$, $i = 1, 2 \dots$.

In such a way we get y^{m+1}. It is easy to see that y^{m+1} and y^m differ possibly only at $[w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]$, $i = 1, 2, \dots$. Thus

$$F_{m+1} = \{i \in \mathbb{Z}_+ : y^{m+1}(i) = 1\} \subset F_m \bigcup \cup_{i=1}^{\infty} [w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1].$$

At the same time B_i appears in y^{m+1} syndetically with gaps bounded by l_i for each $1 \le i \le m+1$ by the construction and so is A_i for each $1 \le i \le m+1$.

In such a way for each $m \in \mathbb{N}$ we defined a finite word A_m. Let $y = \lim A_m = \lim y^m$. By the construction, A_m appears in y with gaps bounded by l_m for each $m \in \mathbb{N}$. That is, y is a minimal point for the shift. It is obvious that $y \neq (0, 0, \dots)$. Let $Y = \overline{orb(y, \sigma)}$ and $U = \{x \in Y : x(0) = 1\}$. Then

$$\emptyset \neq N(y, U) = \bigcup_{i=1}^{\infty} \{i \in \mathbb{Z}_+ : A_n(i) = 1, 0 \le i \le k_n - 1\} \subset \bigcup_{i=1}^{\infty} F_n \subset F.$$

Thus F contains the m-set $N(y, U)$. $\square$

Remark 4.8. In fact, in the proof of the above propostion, we can get furthermore that (Y, σ) is a weak mixing system. Indeed, for each $m \in \mathbb{N}$, A_{m+1} has the form $A_m V_m B_m$ i.e. the form $A_m V_m A_m A_m 0 A_m$, so we know that $N([A_m], [A_m]) = N(y, [A_m]) - N(y, [A_m]) \supset \{a_m, a_m + 1\}$ which implies that Y is weakly mixing (see Lemma 4.9).

4.3. **Condition in** [19]**.** In this subsection we will show that there is no minimal t.d.s. satisfying the sufficient condition in [19, Theorem 3.1]. A t.d.s. (X, T) satisfies

($\star$) *if there is $x \in X$ such that for each neighborhood V of x, there exists $n = n(V)$ such that if $S \subset \mathbb{Z}_+$ is a finite subset with $|s - t| \ge n$ for all distinct $s, t \in S$, then there exists $\ell \in \mathbb{Z}_+$ such that $T^{s+\ell} x \in V$ for all $s \in S$.*

We will show that if (X, T) is a transitive system satisfying ($\star$) then it is weakly mixing. Note that the orbit closure of an $\mathcal{F}_s$-PR point need not be weakly mixing, since if x is $\mathcal{F}_s$-PR and p is a periodic point then (x, p) still is $\mathcal{F}_s$-PR.

First we need a lemma from [24, Lemma 5.1].

Lemma 4.9. *Let (X,T) be a transitive t.d.s.. If for any open non-empty subset U of X there is $s = s_U \in \mathbb{Z}_+$ such that $s, s+1 \in N(U,U)$, then (X,T) is weakly mixing.*

Let $\mathcal{F}_{rs}$ be the smallest family containing $\{n\mathbb{Z}_+ : n \in \mathbb{N}\}$. The following notion was introduced in [24]. Let (X,T) be a t.d.s.. We say (X,T) has *dense small periodic sets*, if for any open and non-empty subset U of X there exist a non-empty closed $A \subset U$ and $k \in \mathbb{N}$ such that A is invariant for T^k. Now we are ready to show

Lemma 4.10. *Let (X,T) be a transitive t.d.s. satisfying $(\star)$. Then (X,T) is weakly mixing, and it has dense small periodic sets.*

Proof. Let U be a non-empty open subset of X and V be a neighborhood of x such that $T^mV \subset U$ for some $m \in \mathbb{N}$. Assume n is the number appearing in the definition of $(\star)$. Then there is $\ell \in \mathbb{Z}_+$ such that $\{\ell+n, \ell+2n, \ell+3n+1\} \subset N(x,V)$. That is, $T^{\ell+n}x, T^{\ell+2n}x, T^{\ell+3n+1}x \in V$, which implies that $T^{m+\ell+n}x, T^{m+\ell+2n}x, T^{m+\ell+3n+1}x \in U$. Thus $\ell+n, \ell+2n, \ell+3n+1 \in N(T^m x, U)$. We have

$$N(U,U) = N(T^m x, U) - N(T^m x, U) \supset \{n, n+1\}.$$

By Lemma 4.9, (X,T) is weakly mixing.

Let x be the point in the definition of $(\star)$ and V be a neighborhood of x with $\overline{V} \neq \emptyset$. For all $k \in \mathbb{Z}_+$ we have $\bigcap_{j=0}^{k} T^{-jn-l(k)}V \neq \emptyset$. That is, $\bigcap_{j=0}^{k} T^{-jn}V \neq \emptyset$. By a compactness argument we have $\bigcap_{j=0}^{\infty} T^{-jn}\overline{V} \neq \emptyset$. This implies that there is $y \in \bigcap_{j=0}^{\infty} T^{-jn}\overline{V}$ such that $T^{jn}y \in \overline{V}$ for all $j \in \mathbb{Z}_+$. Thus, (X,T) has dense small periodic sets. □

With the help of Lemma 4.10 we have

Theorem 4.11. *There is no minimal t.d.s. satisfying $(\star)$.*

Proof. Assume the contrary that there is a minimal t.d.s. (X,T) satisfying $(\star)$. Let x be the point in the definition of $(\star)$ and V be a neighborhood of x with $\overline{V} \neq \emptyset$. By Lemma 4.10 (X,T) has dense small periodic sets, and hence (X,T) is not totally transitive. By Lemma 4.10, (X,T) is totally minimal, a contradiction. □

5. $\mathcal{F}$-product recurrence for zero entropy

Entropy is a measurement of complexity or chaoticity of a t.d.s.. For a t.d.s. (X,T) the entropy of (X,T) will be denoted by $h(T)$. For the basic properties of entropy and how to compute the entropy of a symbolic system we refer [35]. In this section we investigate the properties of points whose product with points with zero entropy is recurrent. The main results are if (x,y) is recurrent for any point y whose orbit closure is a minimal system having zero entropy, then x is $\mathcal{F}_{pubd}$-recurrent, and if (x,y) is recurrent for any point y whose orbit closure is an M-system having zero entropy, then x is minimal. Moreover, it turns out that if (x,y) is recurrent for any recurrent y whose orbit closure has zero entropy, then x is distal.

5.1. $\mathcal{F}$-PR$_0$.

Definition 5.1. Let (X,T) be a t.d.s. and $\mathcal{F}$ be a family. $x \in X$ is $\mathcal{F}$-PR$_0$ if for any t.d.s. (Y,S) and any $\mathcal{F}$-recurrent point $y \in Y$ with $\overline{orb(y,T)}$ zero entropy, (x,y) is a recurrent point of $(X \times Y, T \times S)$.

It is cleat that

$$\begin{array}{ccccccc} \mathcal{F}_{inf}-PR & \longrightarrow & \mathcal{F}_{pubd}-PR & \longrightarrow & \mathcal{F}_{ps}-PR & \longrightarrow & \mathcal{F}_{s}-PR \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{inf}-PR_0 & \longrightarrow & \mathcal{F}_{pubd}-PR_0 & \longrightarrow & \mathcal{F}_{ps}-PR_0 & \longrightarrow & \mathcal{F}_{s}-PR_0 \end{array}$$

Where "$\longrightarrow$" means implication.

Recall that x is $\mathcal{F}_{inf}$-PR if and only if x is distal. We have

Theorem 5.2. *x is $\mathcal{F}_{inf}$-PR$_0$ if and only if it is distal.*

Proof. If x is distal it is clear that it is $\mathcal{F}_{inf}$-PR$_0$. Now assume that x is $\mathcal{F}_{inf}$-PR$_0$.

Let A be an IP-set. Then A contains a sub IP-set B with zero entropy, see for example [22]. Then $N(x,U)$ is IP^* for each neighborhood U. Then x is distal by [15]. □

Similar to Theorem 4.3 we have

Theorem 5.3. *Let (X,T) be a transitive t.d.s. which is disjoint from any minimal system with zero entropy. Then each point in $Tran_T$ is $\mathcal{F}_s$-PR$_0$.*

It was proved in [8] that a transitive diagonal system is disjoint from all minimal t.d.s. with zero entropy. Thus if (X,T) is a transitive diagonal t.d.s. then each transitive point x is in $\mathcal{F}_s$-PR$_0$. The following proposition was proved in [22].

Proposition 5.4. *Every subset of $\mathbb{Z}_+$ with lower Banach density 1 containing $\{0\}$ contains an sm-set whose orbit closure has zero entropy.*

Using the same argument as in Theorem 4.5 we have

Theorem 5.5. *The orbit closure of an $\mathcal{F}_s$-PR$_0$ point is an E-set.*

Proof. Let x be an $\mathcal{F}_s$-PR$_0$ point and U be an open neighborhood of x. If $N(x,U)$ has zero Banach density, then the lower Banach density of $A = \mathbb{Z}_+ \setminus N(x,U)$ is 1. Then by Proposition 5.4, $A \cup \{0\}$ contains $N(y,V)$, where (Y,S) is a minimal set, $y \in Y$, V is an open neighborhood of y and $h(S) = 0$. Thus we have $N((x,y), U \times V) = N(x,U) \cap N(y,V) \subset \{0\}$, a contradiction. □

In [22] a transitive diagonal t.d.s. with unique minimal point was constructed (for the example of topological t.d.s. with unique minimal point, see [21]). Thus we have

$$\mathcal{F}_s - PR_0 \not\Rightarrow \mathcal{F}_s - PR.$$

We remark that there is a minimal x which is $\mathcal{F}_s$-PR$_0$ and is not $\mathcal{F}_s - PR$. In fact by [10] if $h(T) > 0$ then there are asymptotic pairs. By [18] or [26] there are minimal u.p.e. systems.

5.2. $\mathcal{F}_{ps}$**-PR$_0$.** In Theorem 3.4 we have shown that if x is $\mathcal{F}_{ps}$-PR, then x is minimal. Here we are discussing the following question: if x is $\mathcal{F}_{ps}$-PR$_0$, is x minimal? The answer is affirmative. That is we have

Theorem 5.6. *If x is $\mathcal{F}_{ps}$-PR$_0$, then it is minimal.*

Proof. According to the proof of Theorem 3.4 it remains to show that the point y constructed in Proposition 3.3 has zero entropy.

Recall that

$$A_{k+1} = A_k 0^{a_k} A_k A_{k-1}^{n_{k+1}^{k-1}} \dots A_2^{n_{k+1}^2} A_1^{n_{k+1}^1} \le (x_0, \dots, x_{l_{k+1}})$$

with $|A_1|^{n_{k+1}^1} = |A_2|^{n_{k+1}^2} = \dots = |A_{k-1}|^{n_{k+1}^{k-1}} = |A_k|$, a_{k+1} can be divided by $|A_k|$ and $y = \lim_{k \longrightarrow \infty} A_k$. Let $X = \overline{orb(y,\sigma)}$ and $m_k = |A_k|$.

We are going to show that $h(\sigma|X) = 0$. Let

$$B_k(y) = \sharp\{u \in \Omega^k : \exists i \in \mathbb{Z}_+ \text{ such that } u = y[i; i+k-1]\}.$$

Then $h(\sigma|X) = \lim_{k\to\infty} \frac{1}{m_k} \log B_{m_k}(y)$. Let $u \in \{0,1\}^{m_k}$ appear in y. Then $\exists i > k$ such that u appears in A_i. By the way of the construction of $A_j, j \in \mathbb{N}$, it is known that $A_i = W_0 W_1 \cdots W_s$ where W_j has the form of $0^{m_k}, A_k, A_{k-1}^{n_{k+1}^{k-1}}, \dots, A_2^{n_{k+1}^2}, A_1^{n_{k+1}^1}$ with $|0^{m_k}| = |A_k| = |A_{k-1}^{n_{k+1}^{k-1}}| = \dots = |A_2^{n_{k+1}^2}| = |A_1^{n_{k+1}^1}|$. So we have that

$$B_{m_k}(y) \le (m_k+1)(k+1)k \le (m_k+1)^3.$$

It follows that

$$h(\sigma|X) = \lim_{k\to\infty} \frac{1}{m_k} \log B_{m_k}(y) = 0.$$

This ends the proof. □

5.3. **Summary and some questions.** Let $\mathfrak{E}_0$ be the collection of all E-systems with zero entropy, and $\mathfrak{M}_0$ be the collection of all M-systems with zero entropy. The following proposition is from [22].

Proposition 5.7. *The following statements hold.*

1. *If $X \perp \mathfrak{E}_0$ (i.e. X is disjoint from each element of $\mathfrak{E}_0$), then X is minimal and has c.p.e..*
2. *If X is minimal and for each $\mu \in M(X,T)$, $(X, \mathcal{B}_X, \mu, T)$ is a measurable K-system, then $X \perp \mathfrak{E}_0$.*
3. *If X is a minimal diagonal system then $X \perp \mathfrak{M}_0$.*

Thus we have

Theorem 5.8. *The following statements hold.*

1. $\mathcal{F}_{pubd} - PR_0 \not\Rightarrow \mathcal{F}_{inf} - PR_0$.
2. $\mathcal{F}_{ps} - PR_0 \not\Rightarrow \mathcal{F}_{ps} - PR$.
3. $\mathcal{F}_{pubd} - PR_0 \not\Rightarrow \mathcal{F}_{pubd} - PR$.

Proof. (1) Let (X, T) be a minimal t.d.s. such that there is $\mu \in M(X, T)$ with $(X, \mathcal{B}_X, \mu, T)$ being a measurable K-system. Then each point of X is in $\mathcal{F}_{pubd} - PR_0$. Since in such a system, there exists asymptotic pairs, we have $\mathcal{F}_{pubd} - PR_0 \not\Rightarrow \mathcal{F}_{inf} - PR_0$.

(2) and (3) follow from Proposition 5.7. □

The following question is open:

$$\mathcal{F}_{ps} - PR_0 \not\Rightarrow \mathcal{F}_{pubd} - PR_0?$$

Note that it is open the question if there is a t.d.s. in $M_0^{\perp} \setminus E_0^{\perp}$, see [22].

To sum up we have

$$\begin{array}{ccccccc}
\mathcal{F}_{inf} - PR & \xleftarrow{?} & \mathcal{F}_{pubd} - PR & \xleftarrow{?} & \mathcal{F}_{ps} - PR & \xleftarrow{not} & \mathcal{F}_{s} - PR \\
\updownarrow & & \uparrow{\scriptstyle not} & & \uparrow{\scriptstyle not} & & \uparrow{\scriptstyle not} \\
\mathcal{F}_{inf} - PR_0 & \xleftarrow{not} & \mathcal{F}_{pubd} - PR_0 & \xleftarrow{?} & \mathcal{F}_{ps} - PR_0 & \xleftarrow{not} & \mathcal{F}_{s} - PR_0
\end{array}$$

For minimal systems we have

$$\begin{array}{ccccccc}
\mathcal{F}_{inf} - PR & \xleftarrow{?} & \mathcal{F}_{pubd} - PR & \xleftarrow{?} & \mathcal{F}_{ps} - PR & \xleftarrow{?} & \mathcal{F}_{s} - PR \\
\updownarrow & & \uparrow{\scriptstyle not} & & \uparrow{\scriptstyle not} & & \uparrow{\scriptstyle not} \\
\mathcal{F}_{inf} - PR_0 & \xleftarrow{not} & \mathcal{F}_{pubd} - PR_0 & \xleftarrow{?} & \mathcal{F}_{ps} - PR_0 & \xleftarrow{?} & \mathcal{F}_{s} - PR_0
\end{array}$$

6. Factors and extensions

In this section we investigate product recurrent properties for a family under factors or extensions. In this section we will use some tools from the theory of Ellis semigroup, see [5] for details.

6.1. **Definitions on factors.** A *factor map* $\pi : X \to Y$ between the t.d.s. (X, T) and (Y, S) is a continuous onto map which intertwines the actions; one says that (Y, S) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S), and one refers to π as a factor or an extension. The systems are said to be *conjugate* if π is bijective. An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x_1, x_2) : \pi x_1 = \pi x_2\} = (\pi \times \pi)^{-1}\Delta_Y \subset X \times X$.

An extension $\pi : (X, T) \to (Y, S)$ is called *proximal* if $R_\pi \subset P(X, T)$. Similarly we define *distal* extensions. We define π to be an *equicontinuous* extension if for every $\epsilon > 0$ there is $\delta > 0$ such that $(x, y) \in R_\pi$ and $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \epsilon$, for every $n \in \mathbb{N}$. The extension π is called *almost one-to-one* if the set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is a dense G_δ subset of X.

6.2. Product recurrent properties under factors or extensions.

Proposition 6.1. *Let $\pi : X \longrightarrow Y$ be a factor map. If $x \in R(X,T)$ then $\pi(x) \in R(Y,S)$. Conversely, if $y \in R(Y,S)$ then there is $x \in \pi^{-1}(y) \cap R(X,T)$.*

Proof. Let $y \in R(Y,S)$. Then there is an idempotent u with $uy = y$. Take $x' \in \pi^{-1}(y)$ and set $x = ux'$. Then $\pi(ux') = u\pi(x') = y$. □

Corollary 6.2. *Let (Y,S) be a t.d.s and $y \in Y$ be recurrent. Then for any t.d.s. (X,T), there is $x \in X$ such that (x,y) recurrent.*

Proof. One can get the corollary from Proposition 2.4. Now we check how to get it from Theorem 6.1. It is clear that $X \times Y$ is an extension of Y with Y being the closure of the orbit of y. Then by Theorem 6.1 there is $x \in X$ with (x,y) recurrent. □

Let $\pi : (X,T) \to (Y,S)$ be a factor map. Recall a point $x \in X$ is called π-*distal* if $(x',x) \in P(X,T)$ and $\pi(x') = \pi(x)$ then $x = x'$.

Theorem 6.3. *Let $\mathcal{F}$ be a family, $(X,T),(Y,S)$ be two t.d.s. and $\pi : X \longrightarrow Y$ be a factor map.*

1. *if x is $\mathcal{F}$-PR, then $\pi(x)$ is $\mathcal{F}$-PR.*
2. *if x is π-distal and $y = \pi(x)$ is $\mathcal{F}$-PR, then x is $\mathcal{F}$-PR.*
3. *if $y \in Y$ satisfies $\pi^{-1}(y) = \{x\}$ for some $x \in X$ and y is $\mathcal{F}$-PR, then x is $\mathcal{F}$-PR.*

Proof. (1) Let x be $\mathcal{F}$-PR and X be the orbit closure of x. Assume that z is a $\mathcal{F}$-recurrent point and $Z = \overline{orb(z)}$. Then $\pi \times Id : X \times Z \longrightarrow Y \times Z$ is a factor map. Since the orbit closure of (x,z) is transitive and (x,z) is a recurrent point, it follows that $(\pi(x), z)$ is a recurrent point and thus, $\pi(x)$ is $\mathcal{F}$-PR.

(2) Assume y is $\mathcal{F}$-PR. Let z be a $\mathcal{F}$-recurrent point. Then (y,z) is recurrent, and hence there exists an idempotent u such that $u(y,z) = (y,z)$. Now we have $\pi(ux) = u\pi(x) = uy = y = \pi(x)$ and note that $(x,ux) \in P(X,T)$. Since x is π-distal, we have $ux = x$. Thus $u(x,z) = (x,z)$, i.e. (x,z) is recurrent. Hence x is $\mathcal{F}$-PR.

(3) is a special case of (2). □

Theorem 6.4. *Let $(X,T),(Y,S)$ be t.d.s.*

1. *If (X,T) and (Y,S) have dense sets of minimal points (resp. E-systems, P-systems), then so does $X \times Y$.*
2. *If (X,T) has a measure with full support and (Y,S) has a dense set of recurrent points, then $X \times Y$ has a dense set of recurrent points.*
3. *There are transitive t.d.s. (X,T) and (Y,S) such that $X \times Y$ does not have a dense set of recurrent points.*

Proof. If (X,T) and (Y,S) have dense sets periodic points, or have measures with full support it is clear that so does $X \times Y$.

If X and Y are minimal then there is a minimal point $(x, y) \in X \times Y$. Since $T^n \times S^m : X \times Y \longrightarrow X \times Y$ is a factor map it follows that $(T^n x, S^m y)$ is minimal for each pair $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Thus the set of minimal points in $X \times Y$ is dense. This implies that if X and Y have dense sets of minimal points then so does $X \times Y$.

Now assume that X has a measure with full support and Y has a dense set of recurrent points. By the ergodic decomposition without loss of generality we assume that X is an E-system and Y is transitive. For non-empty open sets $U \subset X$ and $V \subset Y$, pick transitive points $x \in U$ and $Y \in V$. Then

$$N(U\times V, U\times V) = N(U,U)\cap N(V,V) = (N(x,U)-N(x,U))\cap(N(y,V)-N(y,V)).$$

Since $N(x,U) \in \mathcal{F}_{pubd}$, $N(x,U)-N(x,U)$ is an IP^*-set. This implies that $N(U\times V, U \times V)$ is infinite. That is $\Omega(T \times S) = X \times Y$ which implies that the set of recurrent points in $X \times Y$ is dense.

Let F_1 and F_2 be two disjoint thick sets. Let A_1 and A_2 be two IP-sets contained in F_1 and F_2 respectively. Moreover we may assume that A_i is generated by $\{p^i_j\}$ with

$$p^i_{j+1} > p^i_1 + \ldots + p^i_j$$

for all $j \in \mathbb{N}$ and $A_i - A_i \subset F_i$ for $i = 1, 2$. Then we claim $X_1 \times X_2$ does not have a dense set of recurrent points, where $X_i = \overline{orb(1_{A_i}, \sigma)}$.

To verify,

$$\begin{aligned} N([1]_X \times [1]_Y, [1]_X \times [1]_Y) &= N([1]_X, [1]_X) \cap N([1]_Y, [1]_Y) \\ &= (A_1 - A_1) \cap (A_2 - A_2) \subset F_1 \cap F_2 = \emptyset. \end{aligned}$$

□

7. Disjointness and weak disjointness

Let $\mathcal{T}$ be a class of t.d.s. and (X, T) be a t.d.s. If $(X, T) \perp (Y, S), \forall (Y, S) \in \mathcal{T}$, then we denote it by $(X, T) \perp \mathcal{T}$ or $(X, T) \in \mathcal{T}^\perp$, where $\mathcal{T}^\perp = \{(X, T) : (X, T) \perp \mathcal{T}\}$.

Let $\mathcal{M}$ be the class of all minimal t.d.s.; $\mathcal{M}_0$ be the class of all minimal t.d.s. with zero entropy; $\mathcal{M}_{eq}$ be the class of all minimal equicontinuous t.d.s.; and $\mathcal{M}_{wm}$ be the class of all minimal weakly mixing t.d.s.. In [14], Furstenberg asked the question: which t.d.s. is disjoint from $\mathcal{M}$? We extend the question:

Question 7.1. *Which t.d.s. is disjoint from $\mathcal{M}$, $\mathcal{M}_0$, $\mathcal{M}_{eq}$ and $\mathcal{M}_{wm}$? Or determine $\mathcal{M}^\perp$, $\mathcal{M}_0^\perp$, $\mathcal{M}_{eq}^\perp$ and $\mathcal{M}_{wm}^\perp$.*

A related question is about the weak disjointness. In this section we will summary what one knows concerning the above question and give additional new results.

7.1. **Some basic properties on disjointness.** Let $\pi : (X,T) \to (Y,S)$ be an extension between two t.d.s. (X,T) and (Y,S). π is called *minimal* if the only closed invariant subset K of X such that $\pi(K) = Y$ is X itself. Clearly, X is minimal if and only if π is minimal and Y is minimal. More generally, let $\pi : X \to Y$, $\psi : Y \to Z$ be extensions, then $\psi \circ \pi$ is a minimal extension if and only if both ψ and π are minimal extensions.

By definitions it is very easy to have the following observation:

Lemma 7.2. *Let (X,T) be a t.d.s. and let (Y,S) be minimal. Then $(X,T) \perp (Y,S)$ if and only if the projection map $\pi_1 : X \times Y \to X$ is a minimal extension.*

An extension $\pi : X \to Y$ is said to be *semi-distal* if every recurrent point of R_π is minimal.

Lemma 7.3. [2, Theorem 2.14.] *Let $\pi : (X,T) \to (Y,S)$ be a factor map. If X is transitive and π is semi-distal, then π is minimal.*

Since each equicontinuous or distal extension is semi-distal, we have

Corollary 7.4. *Let $\pi : (X,T) \to (Y,S)$ be a factor map. If X is transitive and π is equicontinuous or distal, then π is minimal.*

The following proposition concerns the 'lifting' of disjointness by semi-distal extensions.

Proposition 7.5. *Let (X,T) be a t.d.s. and $\pi : (Y',S') \to (Y,S)$ be a factor map. If π is semi-distal (resp. distal, equicontinuous) and $(X \times Y', T \times S')$ is transitive, then*

$$X \perp Y' \quad \text{if and only if} \quad X \perp Y.$$

Proof. It follows from Lemma 7.2 and Lemma 7.3. □

The following proposition concerns the 'lifting' of disjointness by proximal extensions.

Lemma 7.6. *Let $\pi : (X,T) \to (Y,S)$ be a factor map. If X has a dense set of minimal points and π is proximal, then π is minimal.*

Proof. Let J be a closed invariant subset of X with $\pi(J) = Y$. Let x be a minimal point of X. Since $\pi(J) = Y$, there is $x' \in J$ such that $\pi(x) = \pi(x')$. Now as π is proximal, x, x' are proximal. Hence by minimality of x,

$$x \in \overline{orb(x,T)} \subset J.$$

Since the set of minimal points of X is dense, $J = X$. That is, π is minimal. □

Proposition 7.7. *Let (X,T) be a t.d.s. and $\pi : (Y',S') \to (Y,S)$ be a factor map. If π is proximal and $(X \times Y', T \times S')$ has a dense set of minimal points, then*

$$X \perp Y' \quad \text{if and only if} \quad X \perp Y.$$

Proof. It follows from Lemma 7.2 and Lemma 7.6. □

Finally, we have the following property:

Proposition 7.8. [3] *Disjointness is a residual property, i.e. it is inherited by factors, irreducible lifts and inverse limits.*

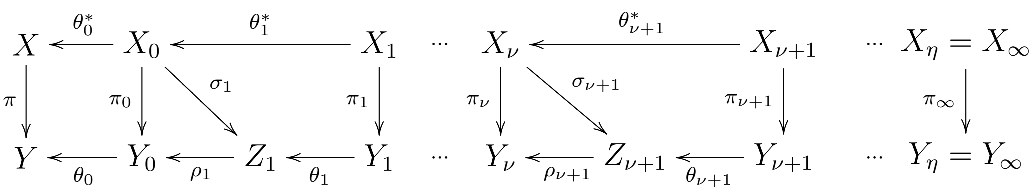
7.2. **Structure theorem for minimal systems.** In this subsection we briefly review the structure theorem of minimal systems. This theory was mainly developed for group actions and accordingly we assume that T is a homeomorphism when we use the related result. To get the results for surjective maps we need to consider the natural extensions.

We say that a minimal system (X,T) is a *strictly PI system* if there is an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota\le\eta}$ such that (i) W_0 is the trivial system, (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous (isometric when X is metrizable), (iii) for a limit ordinal $\nu \le \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$, and (iv) $W_\eta = X$. We say that (X,T) is a *PI-system* if there exists a strictly PI system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost one-to-one extensions (or by highly proximal extensions in the non-metric case) we get the notion of HPI *systems*. If we replace the proximal extensions by trivial extensions (i.e. we do not allow proximal extensions at all) we have I *systems*. These notions can be easily relativize and we then speak about I, HPI, and PI extensions.

We have the following structure theorem for minimal systems, for details see [13, 31] etc.

Theorem 7.9 (Structure theorem for minimal systems)**.** *Given a homomorphism $\pi : X \to Y$ of minimal dynamical system, there exists an ordinal η (countable when X is metrizable) and a canonically defined commutative diagram (the canonical PI-Tower)*



where for each $\nu \le \eta, \pi_\nu$ is RIC, ρ_ν is isometric, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$.

Thus if Y is trivial, then X_∞ is a proximal extension of X and a RIC weakly mixing extension of the strictly PI-system Y_∞. The homomorphism π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if X is a PI-system.

Reall an extension $\pi : X \to Y$ of minimal systems *relatively incontractible (RIC) extension* if it is open and for every $n \ge 1$ the minimal points are dense in the relation

$$R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \le i \le j \le n\}.$$

7.3. **Disjointness for $\mathcal{M}_{pi}$.** In this subsection we discuss disjointness for $\mathcal{M}_{pi}$ which is the collection of all minimal PI-systems. It is known among minimal t.d.s. that $\mathcal{M}_{eq}^{\perp} = \mathcal{M}_{wm}$ which implies that $\mathcal{M}_{pi}^{\perp} = \mathcal{M}_{wm}$. In this subsection we will show that a weakly mixing t.d.s. with dense minimal points is disjoint from all minimal PI-systems.

Let (X, T) be a minimal weakly mixing t.d.s. and $(Y, S) = (X \times X, T \times T)$. Then (Y, S) is weakly mixing and has a dense set of minimal points. We claim that $(Y, S) \not\perp (X, T)$. In fact $J = \{(x, y, x) : x, y \in X\}$ is a join and it is clear that $J \neq X \times X \times X$. This implies that a weakly mixing system with dense minimal points is not necessarily disjoint from all minimal t.d.s.. The following theorem says that under the same assumption we have the disjointness from all minimal PI-systems. We remark that a weakly mixing t.d.s. (even scattering) is disjoint from all HPI minimal t.d.s. (using Propositions 7.5 and 7.8).

Theorem 7.10. *A weakly mixing t.d.s. with dense minimal points is disjoint from all minimal PI-systems.*

Proof. Since a PI system is constructed by equicontinuous and proximal extensions, the result follows from Propositions 7.5, 7.7 and 7.8 and the well known facts:

- a weakly mixing t.d.s., is weakly disjoint from all minimal t.d.s. [9], (since a weakly mixing t.d.s. is scattering).
- the product of two systems with dense sets of minimal points still have a dense set of minimal points (Theorem 6.4).
- a weakly mixing t.d.s. is disjoint from all minimal equicontinuous t.d.s. [14].

□

We have the following remark.

Remark 7.11. By the structure theorem of a minimal t.d.s. and the result in [24] to obtain the necessary and sufficient condition for disjointness from all minimal t.d.s. (for a transitive t.d.s.) is equivalent to find such a condition (implying weakly mixing, dense minimal points and something more) such that if X satisfies the condition, and X is disjoint from a minimal t.d.s. Y', then X is disjoint from all minimal t.d.s. Y satisfying that $\pi : Y \to Y'$ is a weakly mixing extension.

We think that the following question has an affirmative answer.

Question 7.12. *Assume (X, T) is transitive and $(X, T) \in \mathcal{M}_{pi}^{\perp}$. Is it true that (X, T) is a weakly mixing E-system?*

The difficulty answering the question is that we do not know if each subset of $\mathbb{N}$ having lower Banach density 1 and containing 0 contains a minimal point whose orbit closure is a PI system (there is a such set which does not contains a minimal point whose orbit closure is a HPI system, since otherwise we have scattering implies weak mixing).

7.4. **Disjointness and weak disjointness for $\mathcal{M}$.** In [24] it was shown that a weakly mixing system with a dense set of regular minimal points is disjoint from any minimal t.d.s. Now we improve the result by showing that a weakly mixing system with a dense set of distal points is disjoint from any minimal t.d.s. We give two proofs, where the first one is provided by W. Huang and the second one relies on the structure theorem for minimal systems. After that we will give another result on disjointness: each $\mathcal{F}_s$-independent t.d.s. is disjoint from any minimal t.d.s.

First we will prove

Theorem 7.13. *A weakly mixing system with a dense set of distal points is disjoint from any minimal t.d.s..*

To prove it we need a proposition concerning proximal cell, see [4, 23]. Note that for a t.d.s. (X,T) and $x \in X$, $P[x]$ denotes the *proximal cell*, i.e. $P[x] = \{y \in X : y \text{ is proximal to } x\} = \{y \in X : (x,y) \in P(X,T)\}$.

Lemma 7.14. *Let (X,T) be a weakly mixing t.d.s. Then for each $x \in X$, $P[x]$ is a dense G_δ subset of X.*

Proof of Theorem 7.13: Let (X,T) be a weakly mixing system with a dense set of distal points and $\{x_s\}_{s=1}^\infty$ be a dense set of distal points. By Lemma 7.14 there is $x \in \cap_{s=1}^\infty P[x_s]$. Let (Y,S) be a minimal t.d.s. and $J \subset X \times Y$ be a join. Then there is $y \in Y$ such that $(x,y) \in J$. For each x_s, (x,x_s) is proximal, thus for each $\epsilon > 0$,

$$\{n \in \mathbb{Z}_+ : d(T^n x, T^n x_s) < \epsilon/2\}$$

is thick. Since x_s is a distal point, (x_s, y) is minimal and hence

$$\{n \in \mathbb{Z}_+ : d(T^n x_s, x_s) < \epsilon/2, d(T^n y, y) < \epsilon\}$$

is syndetic. Thus, for a given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$d(T^n x, T^n x_s) < \epsilon/2, \ d(T^n x_s, x_s) < \epsilon/2, \text{ and } d(T^n y, y) < \epsilon.$$

That is, $d(T^n x, x_s) < \epsilon$ and $d(T^n y, y) < \epsilon$. This implies that $(x_s, y) \in W =: \overline{orb((x,y), T \times S)}$, and thus $X \times \{y\} \subset W \subset J$. It follows that $J = X \times Y$ since (Y,S) is minimal. To conclude, (X,T) is disjoint from (Y,S). □

Now we give the second proof. Since by Theorem 7.10 each weakly mixing system with a dense set of distal points is disjoint from any PI minimal system, by the structure theorem for minimal systems (Theorem 7.9) we need to deal with weakly mixing extensions.

Lemma 7.15. *Let $\pi : Y' \to Y$ be a weakly mixing RIC extension of minimal systems. Then there is a dense G_δ subset $Y_0 \subset Y'$ such that for each $x \in Y_0$, $P[x]$ is dense in the fibre $\pi^{-1}(\pi(x))$.*

Proof. See Appendix. □

The following proposition concerns the "lifting" of disjointness by weakly mixing extensions. Note that each t.d.s. (X,T) has a natural extension (X',T') such that T'

is a homeomorphism. We may assume that all t.d.s. are invertible when considering disjointness, see [24, Proposition 1.1].

Proposition 7.16. *Let (X,T) be a t.d.s. with a dense set of distal points and let $\pi:(Y',S')\to(Y,S)$ be a weakly mixing RIC extension of minimal systems. Then*

$$X\perp Y' \quad \text{if and only if} \quad X\perp Y.$$

Proof. It suffices to show if $X\perp Y$ then $X\perp Y'$. Let $J\subset X\times Y'$ be a join of X and Y'. Let x be a distal point of X and $y\in\pi(Y_0)$, where Y_0 is defined in Lemma 7.15. We remark that $\pi(Y_0)$ is residual in Y. Since $X\perp Y$, $id\times\pi(J)=X\times Y$. Thus there is some $y_0\in Y'$ such that $(x,y_0)\in J$ and $\pi(y_0)=y$. Let $y'\in P_{Y'}[y_0]\cap\pi^{-1}(y)$. Then $(x,y_0),(x,y')$ are proximal. Since x is distal, (x,y') is minimal. And this implies that

$$(x,y')\in\overline{orb((x,y_0))}\subset J.$$

By Lemma 7.15, such y' is dense in $\pi^{-1}(y)$. Thus $\{x\}\times\pi^{-1}(y)\subset J$. Since $y\in\pi(Y_0)$ is arbitrary we have $\{x\}\times Y'\subset J$. Finally, by the density of distal points in X, we have $J=X\times Y'$. □

Now Theorem 7.13 follows from the structure theorem (Theorem 7.9), Theorem 7.10 and Proposition 7.16.

To prove another disjointness result we need some notions and results from [21].

Definition 7.17. Let (X,T) be a t.d.s.. For a tuple $\mathbf{A}=(A_1,\dots,A_k)$ of subsets of X, we say that a subset $F\subseteq\mathbb{Z}_+$ is an *independence set* for $\mathbf{A}$ if for any nonempty finite subset $J\subseteq F$, we have

$$\bigcap_{j\in J}T^{-j}A_{s(j)}\neq\emptyset$$

for any $s\in\{1,\dots,k\}^J$.

We shall denote the collection of all independence sets for $\mathbf{A}$ by $\mathrm{Ind}(A_1,\dots,A_k)$ or $\mathrm{Ind}\mathbf{A}$.

Definition 7.18. Let $\mathcal{F}$ be a family, $k\in\mathbb{N}$ and (X,T) be a t.d.s.. A tuple $(x_1,\dots,x_k)\in X^k$ is called an *$\mathcal{F}$-independent tuple* if for any neighborhoods $U_1,\dots,U_k$ of $x_1,\dots,x_k$ respectively, one has $\mathrm{Ind}(U_1,\dots,U_k)\cap\mathcal{F}\neq\emptyset$.

A t.d.s. is said to be *$\mathcal{F}$-independent of order k*, if for each tuple of nonempty open subsets $U_1,\dots,U_k$, $\mathrm{Ind}(U_1,\dots,U_k)\cap\mathcal{F}\neq\emptyset$, and a t.d.s. is said to be *$\mathcal{F}$-independent*, if it is *$\mathcal{F}$-independent* of order k for each $k\in\mathbb{N}$.

It is proved in [21] that an $\mathcal{F}_s$-independent t.d.s. is weakly mixing, has positive entropy and has a dense set of minimal points. Moreover, the following lemma is proved.

Lemma 7.19. *For every minimal subshift $X\subseteq\Sigma_2$, $\mathrm{Ind}([0]_X,[1]_X)$ does not contain any syndetic set.*

An easy consequence of the above lemma is that there is no non-trivial minimal t.d.s. which is $\mathcal{F}_s$-independent. Now we are ready to show

Theorem 7.20. *An $\mathcal{F}_s$-independent of order 2 t.d.s. is disjoint from all minimal t.d.s..*

Proof. Since it is an open question if an $\mathcal{F}_s$-independent pair can be lifted by extension, the proof of [8] can not be applied here directly. We will use some part of the proof in [8] and some results in [21].

Let (X,T) be an $\mathcal{F}_s$-independent t.d.s. and (Y,S) be minimal. Assume the contrary that $X \not\perp Y$. Then there is a joining $J \neq X \times Y$. We may assume that J is minimal, i.e. if J' is a joining and $J' \subset J$ then $J' = J$. For $x \in X$ let $J(x) = \{y \in Y : (x,y) \in J\}$. We claim that there exists $x \in X$ such that $J(x) \cap J(Tx) = \emptyset$.

Now suppose that $J(x) \cap J(Tx) \neq \emptyset$ for all $x \in X$. Let

$$J' = \bigcup_{x \in X} \{x\} \times \big(J(x) \cap J(Tx)\big).$$

It is easy to check that $J' \subset J$ is a joining, and hence by minimality $J' = J$. This implies that $J = X \times Y$, a contradiction. So there exists $x \in X$ such that $J(x) \cap J(Tx) = \emptyset$.

There exist disjoint closed neighborhoods W_0 and W_1 of x and Tx such that $J(W_0) \cap J(W_1) = \emptyset$, since J is closed and $J(x) \cap J(Tx) = \emptyset$. So there is an syndetic subset $S \in \mathrm{Ind}(W_0, W_1)$. Let $\pi_X : J \to X$ and $\pi_Y : J \to Y$ be the projections. It is clear that $S \in \mathrm{Ind}(\pi_X^{-1}(W_0), \pi_X^{-1}(W_1))$ and $S \in \mathrm{Ind}(\pi_Y \pi_X^{-1}(W_0), \pi_Y \pi_X^{-1}(W_1))$. Since $J(W_0) \cap J(W_1) = \emptyset$ we know that $\pi_Y \pi_X^{-1}(W_0) \cap \pi_Y \pi_X^{-1}(W_1) = \emptyset$. Let V_0 and V_1 be disjoint closed neighborhoods of $\pi_Y \pi_X^{-1}(W_0)$ and $\pi_Y \pi_X^{-1}(W_1)$ respectively. It is clear that $S \in \mathrm{Ind}(V_0, V_1)$.

It is well known that we can find a minimal t.d.s. (X_1, T_1) and a factor map $\pi : (X_1, T_1) \to (Y, S)$ such that X_1 is a closed subset of a Cantor set. It is easy to see that $\mathrm{Ind}(V_0, V_1) = \mathrm{Ind}(\pi^{-1}(V_0), \pi^{-1}(V_1))$. Write X_1 as the disjoint union of clopen subsets U_0 and U_1 such that $U_j \supseteq \pi^{-1}(V_j)$ for $j = 0, 1$. Then $\mathrm{Ind}(V_0, V_1) \subseteq \mathrm{Ind}(U_0, U_1)$.

Define a coding $\phi : X_1 \to \Sigma_2$ such that for each $x \in X_1$, $\phi(x) = (x_0, x_1, \ldots)$, where $x_i = j$ if $T_1^i(x) \in U_j$ for all $i \in \mathbb{Z}_+$. Then $Z = \phi(X_1)$ is a minimal subshift contained in Σ_2 and $\phi : X_1 \to Z$ is a factor map. It is easy to verify that $\mathrm{Ind}(U_0, U_1) \subseteq \mathrm{Ind}([0]_Z, [1]_Z)$.

By Lemma 7.19 we know that $\mathrm{Ind}([0]_Z, [1]_Z)$ does not contain any syndetic set. This contradicts the fact that $S \in \mathrm{Ind}([0]_Z, [1]_Z)$. So X and Y are disjoint. $\square$

We remark that the assumption of $\mathcal{F}_s$-independence can not be weaken significantly, since there exists an $\mathcal{F}_{pd}$-independent t.d.s. with only one minimal point [21]. So combining the result in [24] we have

Proposition 7.21. *The following hold:*

1. *A weakly mixing system with a dense set of distal points is disjoint from any minimal t.d.s.; and an $\mathcal{F}_s$-independent t.d.s. is disjoint from any minimal t.d.s..*

(2) *If (X,T) is transitive and is disjoint from any minimal t.d.s. then (X,T) is weakly mixing and has a dense set of minimal points.*

Recall that a t.d.s. is scattering it is weakly disjoint from $\mathcal{M}$. In [9] the following proposition was proved:

Proposition 7.22. *A t.d.s. is scattering if and only if for any non-trivial open cover $\mathcal{U}$, $N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) \to \infty$, where a cover is non-trivial if each element of the cover is not dense in X.*

7.5. **Disjointness and weak disjointness for $\mathcal{M}_{eq}$.** Recall that a t.d.s. is *weakly scattering* if it is weakly disjoint from $\mathcal{M}_{eq}$. The following proposition is known, see for example [3].

Proposition 7.23. *A transitive t.d.s. is disjoint from $\mathcal{M}_{eq}$ if and only if it is weakly scattering.*

Let (X,T) and (Y,S) be two transitive t.d.s.. If there exists a continuous map $\phi : Tran_T(X) \to Tran_S(Y)$ with $\phi(Tx) = S\phi(x)$ for $x \in Tran_T(X)$, then we say ϕ is a *generic homomorphism* from (X,T) to (Y,S), (Y,S) is a *generic factor* (X,T) and (X,T) is a *generic extension* of (Y,S). It is not hard to see that if (X,T) is minimal and $\phi : (X,T) \to (Y,S)$ is a generic homomorphism then ϕ is a factor map.

In [27] the authors considered weakly scattering t.d.s.. The following proposition was a result in [27] combing with a simple observation.

Proposition 7.24. *The following hold.*

(1) *A transitive t.d.s. is weakly scattering if and only if it has no non-trivial generic equicontinuous factors.*
(2) *A minimal t.d.s. is disjoint from $\mathcal{M}_{eq}$ if and only if it is weakly mixing.*

Proof. (1) was proved in [27]. To show (2) note that if a minimal t.d.s. is disjoint from $\mathcal{M}_{eq}$ then the maximal equicontinuous factor of (X,T) is trivial, which implies that (X,T) is weakly mixing. There are several ways to show a weakly mixing t.d.s. is disjoint from $\mathcal{M}_{eq}$, say, for example [9, 14]. $\square$

It is clear scattering implies weak scattering. To end the subsection we recall an open question:

Question 7.25. *Does weak scattering implies scattering?*

7.6. **Disjointness for $\mathcal{M}_0$.** As we said it is an open question if weak scattering and scattering are equivalent properties. So in this and the next subsections we only consider disjointness. Let $E(X,T)$ be the set of entropy pairs of (X,T). Recall that a t.d.s. is *diagonal* if $(x,Tx) \in E(X,T)$ for each $x \in X$, and (X,T) has *c.p.e.* if the t.d.s. induced by the smallest closed invariant equivalence relation containing $E(X,T)$ is the trivial one. The following proposition was proved in [22].

Proposition 7.26. *The following statements hold:*

(1) *if a transitive $(X,T) \perp \mathcal{M}_0$ then it is weakly mixing and is an E-system.*

(2) *if* (X,T) *is a transitive diagonal t.d.s. then* $(X,T)\perp\mathcal{M}_0$.
(3) *if* (X,T) *is minimal and* $(X,T)\perp\mathcal{M}_0$ *then* (X,T) *has c.p.e.; and if* (X,T) *is minimal and diagonal, then* $(X,T)\perp\mathcal{M}_0$.

7.7. **Disjointness for** $\mathcal{M}_{wm}$**.** Let $\mathcal{M}_d$ be the collection of all distal systems then $\mathcal{M}_d^\perp=\mathcal{M}_{eq}^\perp\supset\mathcal{M}_{wm}$ which implies that $\mathcal{M}_{wm}^\perp\supset\mathcal{M}_d$. The following proposition is known.

Proposition 7.27. [7] *A minimal t.d.s. is in* $\mathcal{M}_{wm}^\perp$ *if and only if every non-trivial quasi-factor of* X *has a non-trivial distal factor.*

Recall that a *quasi-factor* of X is a minimal subset of $(2^X,T)$, where 2^X is the collection of all non-empty closed subsets of X equipped with the Hausdorff metric.

Definition 7.28. A minimal point x is a *quasi-distal point* if (x,y) is minimal for every minimal y who's orbit closure is weakly mixing.

It is clear that a distal point is quasi-distal. Moreover, if (X,T) is minimal and $(X,T)\in\mathcal{M}_{wm}^\perp$ then each point of X is quasi-distal, since two minimal t.d.s. are disjoint then the product is minimal. By [17, Theorem 2.2], there exists a quasi-distal point which is not distal. Since any almost one-to-one extension of a minimal equicontinuous systems is in $\mathcal{M}_{wm}^\perp$ (say the Denjoy minimal t.d.s.), it follows that there is a quasi-distal point which is not weakly product recurrent. It is not clear if a minimal weakly product recurrent point is quasi-distal. We have the following theorem.

Theorem 7.29. *Let* (X,T) *be a weakly mixing t.d.s. with dense quasi-distal points, then* $(X,T)\in\mathcal{M}_{wm}^\perp$.

Proof. Apply the proof of Theorem 7.13. □

It is well known that a t.d.s. (X,T) is weakly mixing if and only if $N(U,V)$ is thick. Weiss [33] showed that if $F\subset\mathbb{Z}_+$ is a thick set then there is a weakly mixing t.d.s. $(X,T)\subset(\{0,1\}^{\mathbb{Z}_+},\sigma)$ such that $N([1],[1])\subset F$. Huang and Ye [25] showed that if (X,T) is minimal then (X,T) is weakly mixing if and only if $N(U,V)$ has lower Banach density 1. By Remark 4.8 we have

Lemma 7.30. *Let* $F\subset\mathbb{Z}_+$ *be thickly syndetic. Then there are a minimal weakly mixing* $(X,T)\subset(\{0,1\}^{\mathbb{Z}_+},\sigma)$ *and* $x\in X$ *such that* $N(x,[1])\subset F$.

So in the transitive case we have the following corollary:

Proposition 7.31. *If a transitive* (X,T) *is disjoint from all minimal weakly mixing t.d.s. then it is an* M*-system.*

Since a minimal equicontinuous systems is in $\mathcal{M}_{wm}^\perp$, $(X,T)\in\mathcal{M}_{wm}^\perp$ does not imply weak mixing.

The following question remains open:

Question 7.32. *Is it true that a transitive t.d.s.* (X,T) *is disjoint from any minimal t.d.s. if and only if* (X,T) *is weakly mixing and has a dense set of quasi distal points?*

8. Tables

Table 1. $\mathcal{F}$-product recurrence

$\mathcal{F}$	$\mathcal{F}_{inf}$	$\mathcal{F}_{pubd}$	$\mathcal{F}_{ps}$	$\mathcal{F}_s$
Orbit closure of a $\mathcal{F}$-PR point	minimal distal	minimal	minimal	M-system
Orbit closure of a $\mathcal{F}$-PR_0 point	minimal distal	minimal	minimal	E-system

Table 2. Disjointness and weak disjointness

$\mathfrak{F}$	$\mathcal{M}_{eq}$	$\mathcal{M}_{hpi}$	$\mathcal{M}_{pi}$	$\mathcal{M}_{wm}$	$\mathcal{M}$
Properties of transitive systems in $\mathfrak{F}^\perp$	weak scattering	weak scattering	weak mixing + E-system??	M-system	weak mixing + M-system
Systems in $\mathfrak{F}^\perp$	weak scattering	scattering	weak mixing + M-system	weak mixing + dense quasi-distal points	w.m. + dense distal points; $\mathcal{F}_s$-independent
Minimal systems in $\mathfrak{F}^\perp$	weak mixing	weak mixing	weak mixing	every non-trivial quasi-factor has a non-trivial distal factor	trivial
Systems weakly disjoint from $\mathfrak{F}$	weak scattering	??	??	??	scattering

9. More discussions

9.1. $(\mathcal{F}_1, \mathcal{F}_2)$**-product recurrence.** In this subsection we discuss some generalizations of the notions concerning product recurrence.

Definition 9.1. Let $\mathcal{F}_1, \mathcal{F}_2$ be families and (X, T) be a t.d.s. A point $x \in X$ is called *($\mathcal{F}_1$, $\mathcal{F}_2$)-product recurrent* if (x, y) is $\mathcal{F}_2$-recurrent for any $\mathcal{F}_1$-recurrent point y in some t.d.s (Y, S).

By the definition it is obvious that $\mathcal{F}$-product recurrence is nothing but $(\mathcal{F}, \mathcal{F}_{inf})$-product recurrence. As we have seen in this paper, for a family the property $\mathcal{F}$-PR is very complex. Hence it is more difficult to discuss the general case $(\mathcal{F}_1, \mathcal{F}_2)$-PR. But if we assume $\mathcal{F}_1 = \mathcal{F}_2$, then we can use the results from [6, 12]. To see this, let us recall some notions first.

Now we consider the Stone – Čech compactification of the semigroup $\mathbb{Z}_+$ with the discrete topology. The set of all ultrafilters on $\mathbb{Z}_+$ is denoted by $\beta\mathbb{Z}_+$. Let $A \subset \mathbb{Z}_+$ and define $\overline{A} = \{p \in \beta\mathbb{Z}_+ : A \in p\}$. The set $\{\overline{A} : A \subset \mathbb{Z}_+\}$ forms a basis for the open sets (and also a basis for closed sets) of $\beta\mathbb{Z}_+$. Under this topology, $\beta\mathbb{Z}_+$ is the *Stone – Čech compactification* of $\mathbb{Z}_+$.

For $F \subset \mathbb{Z}_+$ the *hull of* F is $h(F) = \overline{F} = \{p \in \beta\mathbb{Z}_+ : F \in p\}$. For a family $\mathcal{F}$, the *hull of* $\mathcal{F}$ is defined by

$$h(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} h(F) = \bigcap_{F\in\mathcal{F}} \overline{F} = \{p \in \beta\mathbb{Z}_+ : \mathcal{F} \subseteq p\} \subseteq \beta\mathbb{Z}_+.$$

Let X be a compact metric space and S a semigroup. Let $\Phi : S \times X \to X$ be an action, i.e. for any $p, q \in S$, $\Phi^p \circ \Phi^q = \Phi^{pq}$. For $(p, x) \in S \times X$, denote

$$px = \Phi(p, x) = \Phi^p(x) = \Phi_x(p).$$

$\Phi^{\#} : S \to X^X$ is defined by $p \mapsto \Phi^p$. Hence $px = \Phi^{\#}(p)(x)$. An *Ellis semigroup* S is a compact Hausdorff semigroup such that the right translation map $R_p : S \longrightarrow S$, $q \longmapsto qp$ is continuous for every $p \in S$. An *Ellis action* of an Ellis semigroup S on a space X is a map $\Phi : S \times X \to X$ which is an action such that the adjoint map $\Phi^{\#}$ is continuous, or equivalently, Φ_x is continuous for each $x \in X$.

Now let (X, T) be a t.d.s. Then $\Phi : \mathbb{Z}_+ \times X \to X, (n, x) \mapsto T^n x$ is an action and it can be extended to an Ellis action $\Phi : \beta\mathbb{Z}_+ \times X \to X$. Hence we have a continuous map $\Phi^{\#} : \beta\mathbb{Z}_+ \to X^X$.

Define

$$H(\mathcal{F}) = H(X, \mathcal{F}) = \Phi^{\#}(h(\mathcal{F})) \subset X^X.$$

It is easy to see that for a family $\mathcal{F}$, $H(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{F}$ has finite intersection property. Moreover, let (X, T) be a t.d.s and $\mathcal{F}$ be a filter. Then $H(\mathcal{F}) = \bigcap_{F\in\mathcal{F}} \overline{T^F} \subseteq X^X$.

Now we generalize the notion of ω-limit set. Let (X, T) be a t.d.s and $\mathcal{F}$ be a family. Define

$$\omega_{\mathcal{F}}(x, T) = \bigcap_{F\in k\mathcal{F}} \overline{T^F(x)},$$

where $T^F = \bigcup\{T^n | n \in F\}$. It is easy to show that if $\mathcal{F}$ is a filter, then $\omega_{k\mathcal{F}}(x, T) = H(\mathcal{F})x$. By the definition one has that a point $x \in X$ is $\mathcal{F}$-recurrent if and only if $x \in \omega_{\mathcal{F}}(x, T)$.

Now let $\mathcal{F}$ be a filterdual (i.e. its dual is a filter). Then a point x is $(\mathcal{F}, \mathcal{F})$-product recurrent if and only if $(x, y) \in \omega_{\mathcal{F}}((x, y), T \times S)$ for any y of some t.d.s. (Y, S) with satisfying $y \in \omega_{\mathcal{F}}(y, S)$. That is, x is $H(\mathcal{F}^*)$-product recurrent defined in [6]. Thus we can use the results in [6, 12] to study $(\mathcal{F}, \mathcal{F})$-PR points.

9.2. **Questions.** We restate the following question in [19].

Question 9.2. *Is each weakly product minimal point distal?*

We conjecture that the above question has a negative answer. The next question concerns disjointness.

Question 9.3. *Let* $(X_1, T_1), (X_2, T_2)$ *and* (Y, S) *be t.d.s. If* $(X_1, T_1) \perp (Y, S)$ *and* $(X_2, T_2) \perp (Y, S)$*, then is it true that*

$$(X_1 \times X_2, T_1 \times T_2) \perp (Y, S)?$$

Or for a class $\mathcal{T}$ *of t.d.s., is finite product closed in* $\mathcal{T}^{\perp}$*?*

10. Appendix: Relative proximal cells

In this appendix we study the relative proximal cell for an independent interest, and on the way to do this, we also give a proof of Lemma 7.15. Here we will use some results from the theory of minimal flows. This theory was mainly developed for group actions and accordingly we assume that T is a homeomorphism in this appendix. Much of this work can be done for a general locally compact group actions. We refer the reader to [5, 31, 32] for details.

10.1. **RIM extension.** Let X be a compact metric space and let $M(X)$ be the collection of regular Borel probability measures on X provided with the weak star topology. Then $M(X)$ is a compact metric space in which X is embedded by the mapping $x \mapsto \delta_x$, where δ_x is the dirac measure at x. If $\phi : X \to Y$ is a continuous map between compact metric spaces, then ϕ induces a continuous map $\phi^* : M(X) \to M(Y)$ which extends ϕ, where $(\phi^*\mu)(A) = \mu(\phi^{-1}A)$ for all Borel set $A \subseteq Y$.

Let (X, T) be a t.d.s.. For each $\mu \in M(X)$, define $(T\mu)(A) = \mu(T^{-1}A)$ for all Borel set $A \subseteq X$. Then $(M(X), T)$ is a t.d.s. too. And if $\pi : X \to Y$ is an extension of t.d.s., then $\pi^* : M(X) \to M(Y)$ is also an extension.

An extension $\pi : X \to Y$ of t.d.s. is said to have a *relatively invariant measure* (RIM for short) if there exists a continuous homomorphism $\lambda : Y \to M(X)$ of t.d.s. such that $\pi^* \circ \lambda : Y \to M(Y)$ is just the (dirac) embedding. In other words: π is a RIM extension if and only if for every $y \in Y$ there is a $\lambda_y \in M(X)$ with $\mathrm{supp}\lambda_y \subseteq \pi^{-1}(y)$ and the map $y \mapsto \lambda_y : Y \to M(X)$ is a homomorphism of t.d.s; this map λ is called a *section* for π. Note that $\pi : X \to \{\star\}$ has a RIM if and only if X has an invariant measure if and only if $M(X)$ has a fixed point, where $\{\star\}$ stands for the trivial system. An extension $\pi : X \to Y$ is called *strongly proximal* if for every pair $\mu \in M(X)$ and $y \in Y$ with $\mathrm{supp}\mu \subseteq \pi^{-1}(y)$, a sequence $\{n_i\}$ can be found such that $\lim T^{n_i}\mu$ is a point mass. It is easy to see that each strongly proximal extension is proximal.

Every extension of minimal systems can be lifted to a RIM extension by strongly proximal modifications. To be precise, for every extension $\pi : X \to Y$ of minimal systems there exists a canonically defined commutative diagram of extensions (called the *G-diagram* [16])

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X^{\#} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi^{\#}} \\ Y & \xleftarrow[\tau]{} & Y^{\#} \end{array}$$

with the following properties:

(a) σ and τ are strongly proximal;
(b) $\pi^{\#}$ is a RIM extension;
(c) $X^{\#}$ is the unique minimal set in $R_{\pi\tau} = \{(x, y) \in X \times Y^{\#} : \pi(x) = \tau(y)\}$ and σ and $\pi^{\#}$ are the restrictions to $X^{\#}$ of the projections of $X \times Y^{\#}$ onto X and $Y^{\#}$ respectively.

By a small modification we can assume that $\pi^{\#}$ is an open RIM extension. We refer to [16, 32] for the details of the construction.

10.2. **Relative regionally proximal relation.** Let $\pi : (X, T) \to (Y, T)$ be t.d.s.. For $\epsilon > 0$ let $\Delta_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Then the relative proximal relation is

$$P_\pi = \bigcap_{n=1}^{\infty} \big(\bigcup_{i\in\mathbb{Z}} T^i \Delta_{1/n}\big) \cap R_\pi,$$

and the relative regionally proximal relation is

$$Q_\pi = \bigcap_{n=1}^{\infty} \overline{\big(\bigcup_{i\in\mathbb{Z}} T^i \Delta_{1/n}\big) \cap R_\pi}.$$

For $R \subseteq X \times X$ and $x \in X$, define $R[x] = \{x' \in X : (x, x') \in R\}$. Define

$$U_\pi[x] = \bigcap_{n=1}^{\infty} \overline{\big(\bigcup_{i\in\mathbb{Z}} T^i \Delta_{1/n}\big)[x] \cap \pi^{-1}(\pi(x))}.$$

In other words: $x' \in U_\pi[x]$ if and only if there are sequences $\{x'_i\}$ in $\pi^{-1}(\pi(x))$ and $\{n_i\}$ in $\mathbb{Z}$ such that

$$x'_i \to x' \quad \text{and} \quad (T \times T)^{n_i}(x, x'_i) \to (x, x).$$

It is clear that $P_\pi[x] \subseteq U_\pi[x] \subseteq Q_\pi[x]$. Define

$$U_\pi = \{(x, x') \in R_\pi : x' \in U_\pi[x]\}.$$

The following is an open question [31]:

Question 10.1. *If $\pi : X \to Y$ is an open Bronstein extension (i.e. R_π has a dense set of minimal points), does $U_\pi[x] = Q_\pi[x]$ for all $x \in X$?*

One does not have an answer for this question, but one has the following result.

Proposition 10.2. [29, Theorem 1.5] *Let $\pi : X \to Y$ be a RIM extension of minimal systems with section λ, and let $y \in Y$ be such that* $\mathrm{supp}\lambda_y = \pi^{-1}(y)$. *Then for all $x \in \pi^{-1}(y)$ we have $U_\pi[x] = Q_\pi[x]$.*

The following lemma guarantees that there are lots of such y in Proposition 10.2.

Lemma 10.3. [16, Lemma 3.3] *Let $\pi : X \to Y$ be a RIM extension of minimal systems with section λ. Then there is a residual set $Y_0 \subseteq Y$ such that $y \in Y_0$ implies* $\mathrm{supp}\lambda_y = \pi^{-1}(y)$.

10.3. **Relative proximal cell.** Let (X, T) be a weakly mixing t.d.s. Then for each $x \in X$, the proximal cell $P[x]$ is a dense G_δ subset of X [4, 23] (under the minimality assumption this result was obtained in [15]). Now we consider the relative case. Let $\pi : X \to Y$ be an extension of t.d.s. and $x \in X$. Call $P_\pi[x]$ the *relative proximal cell* of x.

Question 10.4. *If $\pi : X \to Y$ is an open weakly mixing extension of minimal systems, does the relative proximal cell $P_\pi[x]$ is a residual subset of $\pi^{-1}(\pi(x))$ for all $x \in X$?*

We do not have full answer for this question. But we have the following results.

Theorem 10.5. *Let $\pi : X \to Y$ be a weakly mixing and RIM extension of minimal systems. Then there is a residual set $Y_0 \subseteq Y$ such that for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$ we have that $P_\pi[x]$ is residual in $\pi^{-1}(y)$.*

Proof. By Proposition 10.2 and Lemma 10.3, there is a residual set $Y_0 \subseteq Y$ such that for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$ we have $U_\pi[x] = Q_\pi[x]$. Fix such y and x. Now π is weakly mixing, hence $Q_\pi = R_\pi$. Thus $U_\pi[x] = Q_\pi[x] = R_\pi[x] = \pi^{-1}(y)$. Since $U_\pi[x] = \bigcap_{n=1}^\infty \overline{\big(\bigcup_{i\in\mathbb{Z}} T^i\Delta_{1/n}\big)[x] \cap \pi^{-1}(y)}$, we have

$$\overline{\big(\bigcup_{i\in\mathbb{Z}} T^i\Delta_{1/n}\big)[x] \cap \pi^{-1}(y)} = \pi^{-1}(y), \quad \forall n \in \mathbb{N}.$$

Hence

$$P_\pi[x] = \bigcap_{n=1}^{\infty} \big(\bigcup_{i\in\mathbb{Z}} T^i\Delta_{1/n}\big)[x] \bigcap \pi^{-1}(y)$$

is a residual subset of $\pi^{-1}(y)$. □

Applying the above theorem we have

Theorem 10.6. *Let $\pi : X \to Y$ be an extension of minimal systems. If π is weakly mixing and Bronstein (i.e. R_π has a dense set of minimal points), then there is a residual set $Y_0 \subseteq Y$ such that for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$ we have $P_\pi[x]$ is residual in $\pi^{-1}(y)$.*

Proof. To apply Theorem 10.5, we consider G-diagram:

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma\ } & X^{\#} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi^{\#}} \\ Y & \xleftarrow[\ \tau\ ]{} & Y^{\#} \end{array}$$

First we claim that $(\sigma \times \sigma)R_{\pi^\#} = R_\pi$. By the commutativity of the diagram, we have $(\sigma \times \sigma)R_{\pi^\#} \subseteq R_\pi$. Now we show the converse. Since the minimal points of P_π is dense in R_π it is sufficient to show that every minimal point of R_π is an element of $(\sigma\times\sigma)R_{\pi^\#}$. Let $(x_1, x_2) \in R_\pi$ be minimal, then there is a minimal point $(x'_1, x'_2) \in X^\# \times X^\#$ such that $(\sigma \times \sigma)(x'_1, x'_2) = (x_1, x_2)$. Hence $(\pi^\#(x'_1), \pi^\#(x'_2))$ is a minimal point of $Y^\# \times Y^\#$. But $\tau(\pi^\#(x'_1)) = \tau(\pi^\#(x'_2))$ and τ is proximal, and hence we have $\pi^\#(x'_1), \pi^\#(x'_2)$ are proximal. To conclude we have $\pi^\#(x'_1) = \pi^\#(x'_2)$, i.e. $(x'_1, x'_2) \in R_{\pi^\#}$.

Since π is weakly mixing, it can be shown that $\pi^\#$ is also weakly mixing (for example, see [32, VI(3.19)]). Now $\pi^\#$ is weakly mixing and RIM, by Theorem 10.5, there is a residual set $Y_0^\# \subseteq Y^\#$ such that for all $y^\# \in Y_0^\#$ and all $x^\# \in \pi^{-1}(y^\#)$ we have $P_{\pi^\#}[x^\#]$ is residual in $(\pi^\#)^{-1}(y^\#)$. Let $Y_0 = \tau(Y_0^\#)$. Since $Y^\#$ is minimal and hence τ is semi-open, Y_0 is also a residual subset of Y. Let $y \in Y_0$ and $y^\# \in Y_0^\#$ with $\tau(y^\#) = y$. Let $x \in \pi^{-1}(y)$. Since $(\sigma \times \sigma)R_{\pi^\#} = R_\pi$, we have $\sigma((\pi^\#)^{-1}(y^\#)) = \pi^{-1}(y)$. There is some $x^\# \in (\pi^\#)^{-1}(y^\#)$ such that $\sigma(x^\#) = x$. Since $P_{\pi^\#}[x^\#]$ is dense in $(\pi^\#)^{-1}(y^\#)$, $P_\pi[x]$ is dense in $\pi^{-1}(y)$. But $P_\pi[x]$ always is a G_δ subset of $\pi^{-1}(y)$, and hence it is residual in $\pi^{-1}(y)$. The proof is completed. □

Lemma 7.15 is now followed from Theorem 10.6, since each RIC extension is Bronstein.

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