SENSITIVITY AND REGIONALLY PROXIMAL RELATION IN MINIMAL SYSTEMS

SONG SHAO, XIANGDONG YE AND RUIFENG ZHANG

ABSTRACT. A topological dynamical system is n-sensitive, if there is a positive constant such that in each non-empty open subset there are n distinct points whose iterates will be apart from the constant at least for a same moment. The properties of n-sensitivity in minimal systems are investigated. It turns out that a minimal system is n-sensitive if and only if the n-th regionally proximal relation Q_n contains a point whose coordinates are pairwise distinct. Moreover, the structure of a minimal system which is n-sensitive but not $(n+1)$-sensitive $(n \geq 2)$ is determined.

1. INTRODUCTION

The complexity of a topological dynamical system (TDS for short) is a centrum topic of the research since the introducing the term of chaos in 1975 by Li and Yorke [LY], known as Li-Yorke chaos today. The notion of sensitivity is the kernel in the definition of chaos in the sense of Devaney, which says roughly that in each non-empty open subset there are two points whose trajectories are apart from (at least for one moment) a given positive constant. It is known a transitive system is either sensitive or almost equicontinuous, and especially a minimal system is either equicontinuous or sensitive [AAB]. Moreover, a transitive system with a dense set of minimal points is sensitive [GW], and even a non-minimal Banach-transitive system is sensitive [HY].

In [AK] Akin and Kolyada introduced a notion called Li-Yorke sensitivity which is much stronger than sensitivity, and they showed that a weakly mixing TDS is Li-Yorke sensitive. In [CJ] Cadre and Jacob introduced a notion called pairwise sensitivity for measure preserving transformations (mpt). Particularly they proved that a weakly mixing mpt is pairwise sensitive. Recently, Xiong [X] among other things introduced a new notion called n-sensitivity, which says roughly that that in each non-empty open subset there are n distinct points who trajectories are apart from (at least for one moment) a given positive constant pairwisely.

Date: 2006.08.10.

2000 *Mathematics Subject Classification.* Primary: 37B05.

Key words and phrases. sensitivity, n-sensitivity, regionally proximal relation, minimality.

The first and third authors are partially supported by NSFC (10501042), and the second one is partially supported by NSFC (10531010). Both authors are partially supported by Ministry of Education (20050358053).

We aim to study the properties of n-sensitivity, particularly for minimal systems in this paper. It is proved that a minimal system is n-sensitive if and only if the n-th regionally proximal relation contains a point whose coordinates are distinct. Moreover, the structure of a minimal system which is n-sensitive but not $n+1$-sensitive ($n \geq 2$) is determined. In fact, such a minimal system is a finite to one extension of its maximal equicontinuous factor.

2. Preliminary

In the article, integers, nonnegative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ respectively.

By a *topological dynamical system* (TDS for short) one means a pair (X, T), where X is a compact metric space and $T : X \longrightarrow X$ is continuous and surjective. Let $orb(x, T) = \{T^n x : n = 0, 1, 2, \ldots\}$ be the *orbit of* x. Write (X^n, T) for the n-fold product system $(X\times\cdots\times X, T\times\cdots\times T)$, and set $\Delta_n = \{(x, x, \cdots, x) \in X^n : x \in X\}$ and $\Delta^{(n)} = \{(x_1, x_2, \cdots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}$.

A TDS (X, T) is *transitive* if for any two opene (open and non-empty) sets U and V there is some $n \in \mathbb{N}$ such that $U \cap T^{-n}V \neq \emptyset$. It is called *pointed transitive* if there exists some $x_0 \in X$ such that $\overline{orb(x_0, T)} = X$. Such x_0 is called a *transitive point*. It is easy to see that in our setting these two notions are the same and in fact the collection of transitive points forms a dense G_δ set in X. (X, T) is *weakly mixing* if the product system (X^2, T) is transitive. A TDS (X, T) is *minimal* if $\overline{orb(x, T)} = X$ for every $x \in X$, i.e. every point is transitive. A point x is called *minimal* or *almost periodic* if the subsystem $(\overline{orb(x, T)}, T)$ is minimal.

When (X, T) and (Y, T) are TDSs and $\pi : X \to Y$ is a continuous onto map which intertwines the actions, one says that (Y, T) is a *factor* of the system (X, T) and (X, T) is an *extension* of (Y, T).

A TDS (X, T) is *equicontinuous* for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $d(T^n x_1, T^n x_2) < \epsilon$ for every $n \in \mathbb{N}$. A TDS (X, T) is called *distal* if $\inf\limits_{n\in\mathbb{N}} d(T^n x, T^n x') > 0$ whenever $x, x' \in X$ are distinct.

A pair $(x_1, x_2) \in X^2$ in (X, T) is called *proximal* if there exists a sequence $\{n_i\}$ such that $d(T^{n_i} x_1, T^{n_i} x_2) \to \Delta_2$. The subset $P(X, T)$ of X^2 consisting of all proximal pairs is called the *proximal relation* of (X, T). It is easy to see

$$P(X, T) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} T^{-n}\Delta(1/k),$$

where $\Delta(\epsilon) = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$. The *regionally proximal relation* on X is defined by

$$Q(X, T) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} T^{-n}\Delta(1/k)}.$$

It is clear that $(x, x') \in Q(X, T)$ iff there are $\{x_i\}, \{x'_i\}$ in X and $\{n_i\} \subset \mathbb{N}$ with $(x_i, x'_i) \to (x, x')$ and $T^{n_i}(x_i, x'_i) \to \Delta_2$. It is easy to show that (X, T) is equicontinuous iff $Q(X, T) = \Delta_2$ and (X, T) is distal iff $P(X, T) = \Delta_2$.

3. n-sensitivity and total sensitivity

In this section we shall define and discuss the basic properties of n-sensitivity. Moreover, for a minimal system, a characterization of n-sensitivity will be given. We start with some definitions.

A TDS (X, T) is *sensitive dependence on initial conditions*, or *sensitive* for short, if there exists an $\epsilon > 0$ such that for all $x \in X$ and any neighborhood U of x there are some $y \in U$ and $n \in \mathbb{N}$ with $d(T^n x, T^n y) > \epsilon$. A TDS (X, T) which is not sensitive is called *almost equicontinuous*, i.e. there is some equicontinuous point: there exists $x \in X$ with the property that for any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $y \in X$ satisfies $d(x, y) < \delta$ then $d(T^n x, T^n y) < \epsilon$ for all $n \in \mathbb{N}$. It is known for a transitive almost equicontinuous system, the set of equicontinuous points coincides with the set of transitive points [AAB].

Definition 3.1. For a given integer $n \geq 2$ a system (X, T) is said to be *n-sensitive* if there exists an $\epsilon > 0$ such that for any opene set U there are distinct points $x_1, x_2, \cdots, x_n \in U$ and some $m \in \mathbb{N}$ with

$$min\{d(T^m x_i, T^m x_j) : 1 \leq i \neq j \leq n\} \geq \epsilon.$$

Such an $\epsilon > 0$ is called an *n-sensitive constant* of (X, T).

Note that 2-sensitivity is nothing but the sensitivity. Moreover, for any given $n \geq 2$, there exists a minimal system which is n-sensitive but not $n+1$-sensitive. An example will be presented at the end of the section after a characterization of n-sensitivity for a minimal TDS is given.

As a non-weakly mixing minimal system has a non-trivial equicontinuous factor, the factors of an n-sensitive system need not be n-sensitive. However, the n-sensitivity can be lifted up by a semi-open factor map (a map is *semi-open* if the image of an opene subset contains an opene subset).

Proposition 3.2. *Let (X, T) and (Y, T) be TDS and $\pi : X \to Y$ be semi-open. If (Y, T) is n-sensitive for some $n \geq 2$, so is (X, T).*

Proof. Let $\epsilon' > 0$ be an n-sensitive constant for (Y, S). The continuity of T implies that if $d(y_1, y_2) \geq \epsilon'$ and $\pi(x_i) = y_i$ then $d(x_1, x_2) \geq \epsilon$ for some $\epsilon > 0$.

Let U be an opene subset of X. As π is semi-open, $\pi(U)$ contains an opene subset V. Thus, there are distinct points $y_1, \ldots, y_n$ in Y and $m \in \mathbb{N}$ such that $\min\{d(T^m y_i, T^m y_j) : 1 \leq i \neq j \leq n\} \geq \epsilon'$. Let $x_1, \ldots, x_n \in X$ with $\pi(x_i) = y_i$, $1 \leq i \leq n$. Then $\min\{d(T^m x_i, T^m x_j) : 1 \leq i \neq j \leq n\} \geq \epsilon$, i.e. (X, T) is n-sensitive. □

Definition 3.3. (X, T) is *totally sensitive*, if (X, T) is n-sensitive for every $n \geq 2$.

It is easy to see that any weakly mixing system is totally sensitive. One has the following observation.

Proposition 3.4. *Let (X,T) be a TDS and X be locally connected. If (X,T) is sensitive, then it is totally sensitive.*

Proof. Let $\epsilon > 0$ be a sensitive constant, U be an opene subset of X and $n \geq 3$ be fixed. As X is locally connected, there is a connected opene subset $V \subset U$. By the sensitivity of (X,T), there are $x, x' \in V$ and $m \in \mathbb{N}$ such that $d_0 = d(T^m(x), T^m(x')) \geq \epsilon$.

Let $f : T^m V \longrightarrow \mathbb{R}$ with $f(y) = d(T^m x, y)$. As f is continuous and $T^m V$ is connected, $f(T^m x) = 0$ and $f(T^m x') = d_0$, we have $f(T^m V) \supset [0, d_0]$. So there are n distinct points $x'_1 = T^m x, x'_2, \ldots, x'_{n-1}, x'_n = T^m x' \in T^m V$ such that $d(x'_1, x'_i) = \frac{i-1}{n-1} d_0$ for each $i = 1, 2, \ldots, n$. Take $x_1 = x, x_2, \ldots, x_{n-1}, x_n = x' \in V$ with $T^m x_i = x'_i$ we have $\min\{d(T^m x_i, T^m x_j) : 1 \leq i \neq j \leq n\} \geq \frac{\epsilon}{n-1}$. This implies that (X,T) is n-sensitive and hence totally sensitive. □

Definition 3.5. Let (X,T) be a TDS and $x_1, \cdots, x_n \in X$. The tuple $(x_1, \cdots, x_n)$ is *n-regionally proximal* if there are $x_i^n \to x_i, i = 1, 2, \cdots, n$ and $\{n_i\} \subset \mathbb{N}$ such that

$$T^{n_i}(x_1^n, x_2^n, \cdots, x_n^n) \to \Delta_n.$$

Denote the set of all n-regionally proximal tuples by $Q_n^+(X,T)$.

It is easy to see

$$Q_n^+(X,T) = \bigcap \{ \overline{\bigcup_{m=1}^{\infty} T^{-m}\alpha} : \alpha \text{ is a neighborhood of the diagonal in } X^n \}.$$

Hence it is a closed invariant set of X^n. When the map T is a homeomorphism, let $Q_n^-(X,T) = Q_n^+(X, T^{-1})$ and $Q_n(X,T) = Q_n^+(X,T) \cup Q_n^-(X,T)$. It is not difficulty to check that $Q_n(X,T) = Q_n^+(X,T) = Q_n^-(X,T)$ when T is a minimal homeomorphism.

For a TDS (X,T), $x \in X$ is a *recurrent point* if there is strictly increasing sequence $\{n_i\}$ such that $T^{n_i} x \longrightarrow x$. For a transitive TDS (X,T) if x is a transitive point then for each n, the subset

$$\{(T^{k_1} x, \ldots, T^{k_n} x) : (k_1, \ldots, k_n) \in \mathbb{Z}_+^n\}$$

is dense in X^n, thus the set of recurrent points in X^n is dense in X^n.

For a minimal TDS we have the following characterization of n-sensitivity.

Theorem 3.6. *Let (X,T) be a transitive TDS and $n \geq 2$ be given. If (X,T) is n-sensitive, then $Q_n^+(X,T) \setminus \Delta^{(n)} \neq \emptyset$, and if in addition (X,T) is minimal the converse holds.*

Proof. First assume that (X,T) is n-sensitive. Let U_m be an open subset of X with $diam(U_m) < \frac{1}{m}$ for each $m \in \mathbb{N}$. Then there is $\epsilon > 0$ such that for each m there is a recurrent point $(x_1^m, \ldots, x_n^m) \in U_m \times \cdots \times U_m$ and $t_m \in \mathbb{N}$ with $d(T^{t_m} x_i^m, T^{t_m} x_j^m) > \epsilon$

when $i \neq j$. Without loss of generality assume that $T^{t_m} x_i^m \longrightarrow x_i$, $i = 1, \dots, n$. Thus it is clear that $x_1, \dots, x_n$ are distinct and $(x_1, \dots, x_n) \in Q_n^+(X, T)$.

Now assume that (X, T) is minimal and $Q_n^+(X, T) \setminus \Delta^{(n)} \neq \emptyset$ and $(x_1, x_2, \cdots, x_n) \in Q_n^+(X, T) \setminus \Delta^{(n)}$. As $x_1, x_2, \cdots, x_n$ are distinct, there are disjoint closed neighborhoods A_i of x_i, $i = 1, \dots, n$. Put $\delta = \min\{d(A_i, A_j) : i \neq j\}$.

Let U be an opene subset of X. By minimality there is $n_0 \in \mathbb{N}$ with $\bigcup_{i=0}^{n_0} T^{-i} U = X$. Let δ' be a Lebesgue number for the open cover $U, T^{-1}U, \dots, T^{-n_0}U$. By the definition there are $x_i' \in A_i$ and $n_1 \in \mathbb{N}$ with

$$\max_{1 \leq i < j \leq n} \{d(T^{n_1} x_i', T^{n_1} x_j')\} < \delta'.$$

Thus there exists i_0 with $\{T^{n_1} x_1', \dots, T^{n_1} x_n'\} \subset T^{-i_0} U$ for some $0 \leq i_0 \leq n_0$. We may assume that $(x_1', \dots, x_n')$ is a recurrent point. It is clear that $y_i = T^{n_1 + i_0}(x_i') \in U$ for each $i = 1, \dots, n$. As $(x_1', \dots, x_n')$ is a recurrent point, there is n_2 such that $T^{n_2} y_i \in A_i$, $i = 1, \dots, n$. It is clear that

$$d(T^{n_2} y_i, T^{n_2} y_j) \geq d(A_i, A_j) \geq \delta,$$

when $i \neq j$. Thus, (X, T) is n-sensitive. □

We have the following remark.

Remark 3.7. The converse is not true without the assumption of minimality. For example, let (X, T) be a transitive almost equicontinuous system with a unique minimal point which is a fixed point [AAB]. Then one has $Q_2^+(X, T) = X^2$, but (X, T) is not 2-sensitive.

Now we ready to give the example promised at the beginning of the section.

Example 3.8. There exists an n-sensitive but not $(n+1)$-sensitive minimal system for any $n \geq 2$.

Proof. Let $\pi : (X, T) \to (Y, T)$ be an asymptotic extension of minimal systems, where Y is equicontinuous. If $\max\{Card\ \pi^{-1}(y) : y \in Y\} = n$, then $Q_n(X, T) = \{(x_1, x_2, \cdots, x_n) : x_i \in \pi^{-1}(y) \text{ with cardinality } n\}$ and $Q_{n+1}(X, T) \setminus \Delta^{(n+1)} = \emptyset$. That is, (X, T) is an n-sensitive but not $(n+1)$-sensitive. One can get such an (X, T) by a substitution of length n over two symbols $\{0, 1\}$. For example, let $S = \{0, 1, \cdots, n-1\}$ and $\theta : S \longrightarrow S^3$ with $0 \longrightarrow 010$, $1 \longrightarrow 020$, $2 \longrightarrow 030$, $\cdots$, $n-1 \longrightarrow 000$. Then the substitution minimal system decided by θ satisfies the condition. See [S] for details. □

4. The structure of a non-totally sensitive minimal system and applications

In this section we will determine the structure of a minimal system which is not totally sensitive (i.e. there is some $n \geq 2$ such that it is n-sensitive but not $(n+1)$-sensitive). Recall that if a minimal system (X, T) is invertible, then we have $Q_n(X, T) = Q_n^+(X, T) = Q_n^-(X, T)$.

For a minimal system (X,T), let $(\widetilde{X},\widetilde{T})$ be the natural extension of (X,T), i.e. $\widetilde{X}=\{(x_1,x_2,\dots)\in\prod_{i=1}^{\infty}X : T(x_{i+1})=x_i,\ i\in\mathbb{N}\}$ (as the subspace of the product space) and $\widetilde{T}(x_1,x_2,\dots)=(T(x_1),x_1,x_2,\dots)$. It is clear that $\widetilde{T}:\widetilde{X}\longrightarrow\widetilde{X}$ is a homeomorphism and $\pi_1:\widetilde{X}\to X$ is semi-open [KST], where π_1 is the projection to the first coordinate.

To prove our main theorem we need a result from [MS].

Lemma 4.1. [MS] *Let (X,T) be an invertible minimal system and $x_1,\dots,x_n\in X$. If $(x_1,\dots,x_n)$ is minimal and $(x_i,x_{i+1})\in Q(X,T)$ for $1\le i\le n-1$, then $(x_1,\dots,x_n)\in Q_n(X,T)$.*

The proof of the above theorem needs the tool of enveloping semigroup associated with a TDS. Given a TDS (X,T) its *enveloping semigroup* $E(X,T)$ is defined as the closure of the set $\{T^n : n=0,1,\dots\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). If $u\in E(X,T)$ with $u^2=u$, then u is called *an idempotent.* A *left ideal* in $E(X,T)$ is a non-empty subset I with $E(X,T)I\subset I$. *A minimal left ideal* is one which does not properly contain a left ideal. An idempotent in a minimal left ideal is said to be a *minimal idempotent.* It is known that $x\in X$ is a minimal point if and only if $ux=x$ for some minimal idempotent u [Au]. Using this result we can prove

Theorem 4.2. *Let (X,T) be an invertible minimal system. If (X,T) is n-sensitive but not $(n+1)$-sensitive for some $n\ge 2$, then it is a finite to one extension of its maximal equicontinuous factor. Moreover, the maximal cardinality of the fibre of this extension is less than n^2+1.*

Conversely, if $\pi:X\longrightarrow X_{eq}$ is the factor map to the maximal equicontinuous factor of (X,T) and $\max_{x\in X_{eq}}Card(\pi^{-1}x)<\infty$, then (X,T) is either equicontinuous or n-sensitive not $n+1$-sensitive for some $n\ge 2$.

Proof. First let (X,T) be a minimal system which is n-sensitive but not $n+1$-sensitive for some $n\ge 2$. Let $\pi:(X,T)\to(X_{eq},T)$ be the maximal equicontinuous factor. Since $Q(X,T)$ is an equivalence relation [Au] and (X_{eq},T) is the factor induced by $Q(X,T)$, $Q(X,T)=R_\pi=\bigcup_{y\in Y}\pi^{-1}y\times\pi^{-1}y$.

Let u be a minimal idempotent and $y\in X_{eq}$. If $Card(u\pi^{-1}(y))>n$, then there are distinct $x_1,\dots,x_{n+1}\in u\pi^{-1}(y)$. It is clear that $(x_1,\dots,x_{n+1})$ is a minimal point and $(x_i,x_{i+1})\in Q(X,T)$ for $1\le i\le n$. By Lemma 4.1, $(x_1,\dots,x_{n+1})\in Q_{n+1}(X,T)\setminus\Delta^{(n+1)}$. By Theorem 3.6 (X,T) is $(n+1)$-sensitive, a contradiction! Thus one has $Card(u\pi^{-1}(y))\le n$.

Now we show $Card(\pi^{-1}(y))\le n^2$. If this is not the case, then there is $X_0\subseteq\pi^{-1}(y)$ such that $Card(X_0)=n+1$ and $u(X_0)$ is a singleton. Let $X_0=\{x_1,x_2,\cdots,x_{n+1}\}$, then it is clear that $(x_1,x_2,\cdots,x_{n+1})$ is in $Q_{n+1}(X,T)\setminus\Delta^{(n+1)}$ by the definition. Again by Theorem 3.6, (X,T) is $(n+1)$-sensitive, a contradiction! Hence π is a finite to one extension of its maximal equicontinuous factor, and the maximal cardinality of the fibre of this extension is less than n^2+1.

The other statement of the theorem follows directly from Theorem 3.6. □

Now we are going to extend the result for any minimal TDS. We need the following lemma.

Lemma 4.3. *Let (X,T) be a minimal TDS and $(\widetilde{X},\widetilde{T})$ be its natural extension. Then (X,T) is not totally sensitive if and only if $(\widetilde{X},\widetilde{T})$ is not totally sensitive.*

Proof. Note that $(\widetilde{X},\widetilde{T})$ is minimal, and (X,T) is equicontinuous if and only if $(\widetilde{X},\widetilde{T})$ is equicontinuous.

Assume that (X,T) is not totally sensitive. If (X,T) is equicontinuous, so is $(\widetilde{X},\widetilde{T})$. So we assume that (X,T) is n-sensitive, but not $(n+1)$-sensitive for some $n \geq 2$. Let

$$m = \max_{x\in X} Card(\pi_1^{-1}x) = \max_{x\in X}\max_{i\in\mathbb{N}} Card(T^{-i}(x)).$$

We claim that $m \leq n$. Otherwise, there are $x \in X$ and $t \in \mathbb{N}$ with $Card(T^{-t}(x)) \geq n+1$. Let $T^{-t}(x) = \{y_1,\dots,y_{k'}\}$ with $k' \geq n+1$. Then it is clear that $(y_1,\dots,y_{k'}) \in Q_{k'}(X,T)$, and thus by Theorem 4.2 (X,T) is $n+1$-sensitive, a contradiction. Hence we have $m \leq n$

If $(\widetilde{X},\widetilde{T})$ is $nm+1$-sensitive, then there are $(\tilde{x}_1,\dots,\tilde{x}_{nm+1}) \in Q_{nm+1}(\widetilde{X},\widetilde{T}) \setminus \Delta^{(nm+1)}$ by Theorem 3.6. It is clear that $(\pi_1\tilde{x}_1,\dots,\pi_1\tilde{x}_{nm+1}) \in Q_{nm+1}(X,T)$. Since the cardinality of $\{\pi_1\tilde{x}_1,\dots,\pi_1\tilde{x}_{nm+1}\}$ is larger than n, this implies that (X,T) is $n+1$-sensitive by Theorem 3.6, a contradiction. So $(\widetilde{X},\widetilde{T})$ is not totally sensitive.

Now assume that $(\widetilde{X},\widetilde{T})$ is not totally sensitive. It is clear by Proposition 3.2 that (X,T) is not totally sensitive, since π_1 is semi-open. □

With the above preparation now we can show the main result of the section.

Theorem 4.4. *Let (X,T) be a minimal system. If (X,T) is n-sensitive but not $(n+1)$-sensitive for some $n \geq 2$, then it is a finite to one extension of its maximal equicontinuous factor. Moreover, the maximal cardinality of the fibre of this extension is less than n^4+1.*

Conversely, if $\pi : X \longrightarrow X_{eq}$ is the factor map to the maximal equicontinuous factor of (X,T) and $\sup_{x\in X_{eq}} Card(\pi^{-1}x) < \aleph_0$, then (X,T) is either equicontinuous or n-sensitive not $n+1$-sensitive for some $n \geq 2$.

Proof. Assume that (X,T) is a minimal TDS. Let $(\widetilde{X},\widetilde{T})$ be the natural extension of (X,T), then $\pi_1 : \widetilde{X} \to X$ is semi-open [KST], where π_1 is the projection to the first coordinate. By Lemma 4.3, $(\widetilde{X},\widetilde{T})$ is not totally sensitive. If $\widetilde{\pi} : \widetilde{X} \longrightarrow \widetilde{X}_{eq}$ is the factor map to the maximal equicontinuous factor, then $\widetilde{X}_{eq}$ is induced by the equivalence relation $\widetilde{Q}(\widetilde{X},\widetilde{T})$.

Define $\pi : X \longrightarrow \widetilde{X}_{eq}$ such that for each $x \in X$, $\pi(x) = \widetilde{\pi}(x')$, where $x' \in \pi_1^{-1}x$. If $x', x'' \in \pi_1^{-1}x$, then it is easy to see $(x',x'') \in \widetilde{Q}(\widetilde{X},\widetilde{T})$. Thus π is well-defined. Moreover, it is easy to check that π is continuous and onto, and is a factor map. This implies that $\widetilde{X}_{eq}$ is the maximal equicontinuous factor of (X,T), since a factor of (X,T) is also a factor of $(\widetilde{X},\widetilde{T})$. By the proof of Lemma 4.3, we know that $m=\max_{x\in X} Card(\pi_1^{-1}(x))\leq n$ and $(\widetilde{X},\widetilde{T})$ is not $nm+1$-sensitive, and hence is not n^2+1-sensitive. Then by Theorem 4.2, $\max_{x\in\widetilde{X}_{eq}} Card(\widetilde{\pi}^{-1}(x))\leq n^4$. So we have $\max_{x\in\widetilde{X}_{eq}} Card(\pi^{-1}(x))\leq n^4$, since $\widetilde{\pi}=\pi\circ\pi_1$. □

By Theorem 4.4 we have the following characterization of total sensitivity.

Theorem 4.5. *Let (X,T) be a minimal TDS and $\pi: X\longrightarrow X_{eq}$ be the factor map to the maximal equicontinuous factor of (X,T). Then (X,T) is totally sensitive if and only if* $\sup_{x\in X_{eq}} Card(\pi^{-1}x)\geq\aleph_0$.

In the spirit of Lemma 4.3 and Theorem 4.4, in the rest of this section we assume that the system considered is invertible for convenience.

Let $\pi:(X,T)\longrightarrow(Y,T)$ be a factor map. π is an *almost one to one* extension if there exists a dense G_δ set $X_0\subseteq X$ such that $\pi^{-1}(\{\pi(x)\})=\{x\}$ for any $x\in X_0$. π is said to be an *isometric extension* or *equicontinuous extension* if for each $\epsilon>0$ there is $\delta>0$ such that if $\pi(x_1)=\pi(x_2)$ and $d(x_1,x_2)<\delta$ then $d(T^n(x_1),T^n(x_2))<\epsilon$ for all $n\in\mathbb{N}$. A minimal TDS is a *HPI-flow* if its some almost one to one extension is an inverse limit space by almost one to one or isometric extensions. A point x is called a *distal point* if there is no other point in its orbit closure which is proximal to it. A TDS (X,T) with some distal point whose orbit is dense in X is called a *point-distal system*. The Veech Structure Theorem said a minimal system is point-distal if and only if it is HPI [Au].

Given a factor map $\pi: X\to Y$ between minimal systems (X,T) and (Y,S) there exists a commutative diagram of factor maps (called *O-diagram*)

$$\begin{array}{ccc} X & \xleftarrow{\sigma^*} & X^* \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi^*} \\ Y & \xleftarrow{\tau^*} & Y^* \end{array}$$

such that
(a) σ^* and τ^* are almost one to one extensions; (b) π^* is an open extension (i.e. it is open as a map); (c) X^* is the unique minimal set in $R_{\pi\tau^*}=\{(x,y)\in X\times Y^*:\pi(x)=\tau^*(y)\}$ and σ^* and π^* are the restrictions to X^* of the projections of $X\times Y^*$ onto X and Y^* respectively. We sketch the construction of these factors. Let $(M,\mathbb{Z})$ be the universal minimal action defined from $\mathbb{Z}$. The set M is a closed semigroup with continuous right translations, isomorphic to any minimal left ideal in $\beta\mathbb{Z}$, the *Stone* $-$ *Čech* compactification of $\mathbb{Z}$. Let 2^X be the collection of nonempty closed subsets of X endowed with the Hausdorff topology. Then $T:2^X\to 2^X$, $A\mapsto TA$ define a TDS $(2^X,T)$. To avoid ambiguities one denotes the action of $\beta\mathbb{Z}$ on 2^X by the *circle operation* as follows: let $p\in\beta\mathbb{Z}$ and $A\in 2^X$, then define $p\circ A=\lim_\lambda m_\lambda A$ for any net $\{m_\lambda\}_{\lambda\in\Lambda}$ converging to p. Let $x\in X$, u idempotent with $ux=x$ and $y=\pi(x)$. Let $y^*=u\circ\pi^{-1}(\{y\})$ and define $Y^*=\{p\circ y^*: p\in M\}$ as the orbit closure of y^* in 2^X; one has that y^* is a minimal point so Y^* is minimal. Finally

$X^* = \{(px, p \circ y^*) \in X \times Y^* : p \in M\}$, $\tau^*(p \circ y^*) = py$ and $\sigma^*((px, p \circ y^*)) = px$. It can be proved that $X^* = \{(\tilde{x}, \tilde{y}) \in X \times Y^* : \tilde{x} \in \tilde{y}\}$. See [Au] for details.

Combining Theorem 4.2 with the construction of O-diagram, an immediate consequence is

Corollary 4.6. *Let (X,T) be a minimal system which is n-sensitive but not $(n+1)$-sensitive for some $n \geq 2$. Then (X,T) is HPI.*

Proof. Now we consider the O-diagram of $\pi : X \to X_{eq}$:

$$\begin{array}{ccc} X & \xleftarrow{\sigma^*} & X^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi^*} \\ X_{eq} & \xleftarrow{\tau^*} & {X_{eq}}^* \end{array}$$

Let $x \in X$, u idempotent with $ux = x$ and $y = \pi(x)$. By the proof of Theorem 4.2, $\pi^{-1}(\{y\})$ is less than $n^2 + 1$ and $y^* = u \circ \pi^{-1}(\{y\}) = u\pi^{-1}(\{y\})$ is less than $n + 1$. So π^* is an isometric extension and every fibre has the same cardinality with y^* which is less than $n+1$. It is easy to verify that any finite to one open extension is isometric (or see [MS] for a proof). Hence π^* is isometric. By the definition of HPI, (X,T) is an HPI system. □

For a TDS (X,T) let $h^s(X,T) = \sup_A h^A(X,T)$, where A ranges over all infinite sequences and $h^A(X,T)$ is the sequence entropy with respect to A (see [Go] for details). It is known that if (X,T) has positive entropy then $h^s(X,T) = \infty$, and generally $h^s(X,T) = \log k$ for some $k = \mathbb{N} \cup \{\infty\}$ [HY1].

Now we can give some conditions implying total sensitivity.

Corollary 4.7. *Let (X,T) be a minimal system. If it satisfies one of the following conditions, then it is totally sensitive.*

(1) *infinite sequence entropy;*
(2) *distal but not equicontinuous;*
(3) *not point-distal, i.e. not HPI.*

Proof. (1). Let (X,T) be a minimal TDS and $\pi : (X,T) \to (X_{eq},T)$ be the maximal equicontinuous factor. If (X,T) is not totally sensitive, then by Theorem 4.2 $\max_{y \in X_{eq}} Card(\pi^{-1}(y)) \leq n^2$ for some $n \in \mathbb{N}$. This implies that for a given sequence A

$$h^A(X,T) \leq h^A(X_{eq},T) + \log n^2 = 2\log n$$

by a well known result in [Go], a contradiction.

(2). The following theorem was proved by Sacker and Sell [SS]: Let $\pi : (X,T) \to (Y,T)$ be an extension of distal minimal systems. If there is some $y_0 \in Y$ with $Card\ \pi^{-1}(y_0) = N$, then (X,T) is equicontinuous iff (Y,T) is. Thus the result follows by this fact and Theorem 4.2.

(3). It follows from Corollary 4.6. □

Concerning isometric extensions, we have the following remark.

Example 4.8. Let $\mathbb{T}$ and $\mathbb{T}^2$ be the one and two dimensional torus respectively. Define $T : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such that $T(z,w) = (\alpha z, zw)$, and $T_\alpha : \mathbb{T} \longrightarrow \mathbb{T}$ with $T_\alpha(z) = \alpha z$, where α is not a root of the unit. Then, $(\mathbb{T}^2, T)$ is minimal, distal and not equicontinuous [Au, p.75]. Let $\pi : \mathbb{T}^2 \longrightarrow \mathbb{T}$ with $\pi(z,w) = z$. Then π is an isometric extension. As $(\mathbb{T}^2, T)$ is not equicontinuous, $(\mathbb{T}^2, T)$ is sensitive and hence totally sensitive by Proposition 3.4. Since $(\mathbb{T}, T_\alpha)$ is equicontinuous, an isometric extension can not keep n-sensitivity.

References

[AAB] E. Akin, J. Auslander and K. Berg, *When is a transitive map chaotic*? Convergence in Ergodic Theory and Probability (Columbus, OH, 1993) (Ohio University Math. Res. Inst. Pub., 5). de Gruyter, Berlin, 1996, pp. 25-40.

[AG] E. Akin and E. Glasner, *Residual properties and almost equicontinuity*, J. d'Anal. Math., **84**(2001), 243-286.

[AK] E. Akin and S. Kolyada, *Li-Yorke sensitivity*, Nonlinearity, **16**(2003), 1421-1433.

[Au] J. Auslander, *Minimal flows and their extensions*, (North-Holland Mathematics Studies, 153). North-Holland, Amsterdam, 1988.

[CJ] B. Cadre and P. Jacob, *On pairwise sensitivity*, J. Math. Anal. Appl., **309**(2005), 375-382.

[E] R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, Inc., New York 1969

[GW] E. Glasner and B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, **6**(1993), 1067–1075.

[Go] T.N.T Goodman, *Topological sequence entropy*, Proc. London Math. Soc. **29**(1974), 331–350.

[HY] W. Huang and X. Ye, *An explicite scattering, non-weakly mixing example and weak disjointness*, Nonlinearity, **15**(2002), 1–14.

[HY1] W. Huang and X. Ye, *Combinatorial lemmas and applications*, preprint, 2006.

[KST] S. Kolyada, L. Snoha and S. Trofimchuk, *Noninvertible minimal maps*, Fund. Math. 168 (2001), No. 2, 141–163.

[LY] T. Li and J. Yorke, *Period 3 implies chaos*, Amer. Math. Monthly, **82**(19750, 985–992.

[MS] A. Maass and S. Shao, *Structure of bounded topological sequence entropy minimal systems and n-sequence entropy tuples*, submitted, 2006.

[S] S. Shao, *Dynamical systems with enveloping semigroup containing finitely many idempotents*, J. China Univ. Sci. Tech. Vol.35, No.1(2005), 12–24.

[SS] R. J. Sacker and G. R. Sell, *Finite extensions of minimal transformation groups*, Trans. Amer. Math. Soc., **190**(1974), 325–334.

[X] J. Xiong, *Chaos in topological transitive systems*, Science in China, **48**(2005), 929-939.

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China

E-mail address: songshao@ustc.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China

E-mail address: yexd@ustc.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P.R. China

E-mail address: rfzhang@mail.ustc.edu.cn