Non-Wandering Sets of the Powers of Maps of a Star

Song SHAO and Xiangdong YE *
Department of Mathematics, University of Science and Technology of China,
Hefei, Anhui, 230026, P.R. China

Abstract Let T be a star and $\Omega(f)$ be the set of non-wandering points of a continuous map $f : T \longrightarrow T$. For two distinct prime numbers p and q, we prove: (1) $\Omega(f^p) \cup \Omega(f^q) = \Omega(f)$ for each $f \in C(T,T)$ if and only if $pq > End(T)$, (2) $\Omega(f^p) \cap \Omega(f^q) = \Omega(f^{pq})$ for each $f \in C(T,T)$ if and only if $p + q \geq End(T)$, where $End(T)$ is the number of the ends of T. Using (1)-(2) and the results in [3], we obtain a complete description of non-wandering sets of the powers of maps of 3-star and 4-star.

1 Introduction

In the study of the dynamics of a continuous map $f : X \longrightarrow X$ of a compact metric space X into itself, a central role is played by the various recursive properties of the points of X([1][4][5]). One of the important such properties is non-wanderingness. It is easy to show that the non-wandering set $\Omega(f)$ is a non-empty closed invariant subset of X, but generally $\Omega(f) = \Omega(f^n)$, $n \in \mathbb{N}$ does not hold. So, it is important to know the interrelations of $\Omega(f^n)$, $n \in \mathbb{N}$.

Coven and Nitecki discussed non-wandering sets of the powers of continuous maps of a compact interval in [2]. For a continuous map $f : I \longrightarrow I$ of the interval, they proved:

(1) If there is some $n \geq 2$ such that $x \in \Omega(f) \setminus \Omega(f^n)$, then $Orb(x, f)$ is finite and the topological entropy $h(f) > 0$.

(2) $\Omega(f) = \Omega(f^n)$ whenever n is odd.

Wen Huang and Xiangdong Ye generalized the above results from a compact interval to a tree [3], and the method they used is different from [2]. More precisely, they showed:

1. Let $f : T \longrightarrow T$ be a continuous map of a tree T. If there is some $n \geq 2$ such that $x \in \Omega(f) \setminus \Omega(f^n)$, then $Orb(x, f)$ is finite and the topological entropy $h(f) > 0$.
2. Let T be a tree and $k, n \in \mathbb{N}$. Then $\Omega(f^k) = \Omega(f^{kn})$ for each $f \in C(T,T)$ if and only if n is $(T, k)-$admissible (see [3] for the details).

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In [2] the authors used their results to give a nice description of non-wandering sets of the powers of continuous maps of the interval. More precisely, they showed:

a. For any continuous map f of the interval, all the possible sets $\Omega(f^n),\ n \in \mathbb{N}$ appear in the nested sequence

$$\Omega(f) \supseteq \Omega(f^2) \supseteq \Omega(f^4) \supseteq \dots . \qquad (*)$$

b. Any pre-assigned sequence of equalities and strict containments in $(*)$ can be realized by some continuous map of the interval.

In this paper, we study the interrelations of non-wandering sets of the powers of maps of a star and obtain the following results:

Theorem 1 Let T be a star and p, q be two distinct prime numbers, then $\Omega(f^p) \cup \Omega(f^q) = \Omega(f)$ for each $f \in C(T, T)$ if and only if $pq > End(T)$.

Theorem 2 Let T be a star and p, q be two distinct prime numbers, then $\Omega(f^p) \cap \Omega(f^q) = \Omega(f^{pq})$ for each $f \in C(T, T)$ if and only if $p + q \geq End(T)$.

Combining Theorems 1-2 and the results in [3] we have

Theorem 3 Assume that T is a 3-star or a 4-star.

1.1. For any continuous map f of T, all the possible sets $\Omega(f^n),\ n \in \mathbb{N}$ appear in the following graph (we call it graph A in the sequel, and use $\longrightarrow$ and $\downarrow$ to indicate $\supseteq$).

$$\begin{array}{ccccccccccc} \Omega(f) & \rightarrow & \Omega(f^2) & \rightarrow & \Omega(f^{2^2}) & \rightarrow & \Omega(f^{2^3}) & \rightarrow & \Omega(f^{2^4}) & \rightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \dots \\ \Omega(f^3) & \rightarrow & \Omega(f^{3\cdot 2}) & \rightarrow & \Omega(f^{3\cdot 2^2}) & \rightarrow & \Omega(f^{3\cdot 2^3}) & \rightarrow & \Omega(f^{3\cdot 2^4}) & \rightarrow & \dots \end{array}$$

1.2. For any continuous map f of T,

a. $\Omega(f^{2^i}) \setminus \Omega(f^{2^{i+1}}) = \Omega(f^{3\cdot 2^i}) \setminus \Omega(f^{3\cdot 2^{i+1}})$ for each $i \in \{0\} \cup \mathbb{N}$,

b. $\Omega(f^{2^i}) \setminus \Omega(f^{3\cdot 2^i}) = \Omega(f) \setminus \Omega(f^3)$, for each $i \in \mathbb{N}$.

2. Any pre-assigned sequence of equalities and strict containments in the first line and the first column of the graph A can be realized by some continuous map of T.

We remark that for $n-$star ($n \geq 5$) other graphs should be introduced.

2 Definitions and Elementary Properties

By a *graph* we mean a connected compact one-dimensional polyhedron. A *tree* is a graph without any subset which is homeomorphic to the unit circle. For a given tree T, a *subtree* of T is a subset of T which is a tree itself. For $x \in T$ the number of connected components of $T \setminus \{x\}$ is called the *valence* of T, and if the number is n then we write $Val(x) = n$. A point of T of valence 1 is called an *end* of T, and a point of valence different from 2 is called a *vertex* of T. The set of ends of T, the set of vertices of T and the number of the ends of T will be denoted by $E(T)$, $V(T)$ and $End(T)$ respectively. The closure of each connected component of $T \setminus V(T)$ is called an *edge*.

A *star* is either a tree having only one vertex with valence larger than 2 or an arc. Set $Int(T) = T \setminus E(T)$. Let A be a subset of T containing at least two points. By $[A]$ we denote the convex hull of A in T. If $A = \{a, b\}$ then we use $[a, b]$ to denote $[A]$. We define $(a, b) = [a, b] \setminus \{a, b\}$, and similarly we define $[a, b)$ and $(a, b]$.

Let T be a tree, the collection of all continuous maps from T into itself will be denoted by $C(T, T)$. For $f \in C(T, T)$ and $x \in T$, $\{x, f(x), f^2(x), \ldots\}$ is called the *orbit* of x and is denoted by $Orb(x, f)$. x is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$. Let $P(f)$ denote the set of periodic points of f. $x \in T$ is *non-wandering* if for every neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for some $n \in \mathbb{N}$. The set of non-wandering points of f is denoted by $\Omega(f)$.

To prove our results we need the following lemmas. The first two lemmas are routing generalization of the corresponding results for the interval maps (see [5]) and Lemmas 3-4 are easy to prove.

Lemma 1 Let T be a tree, and $f \in C(T, T)$ and U be a connected subset of T. If there is $n \in \mathbb{N}$ with $f^n(U) \cap U \neq \emptyset$, then $K = U \cup f(U) \cup f^2(U) \cup \ldots$ has finitely many connected components.

Proof. As f is continuous and U is connected, $f^k(U), k \in \mathbb{N}$ are all connected. Since $f^n(U) \cap U \neq \emptyset$, for any $i \in \{0, 1, \cdots, n-1\}$, $f^{i+jn}(U) \cap f^{i+(j+1)n}(U) \neq \emptyset, j = 0, 1, 2, \cdots$. Hence for any $i \in \{0, 1, \cdots, n-1\}$, $K_i = f^i(U) \cup f^{i+n}(U) \cup f^{i+2n}(U) \cup \cdots$ is connected. So $K = K_0 \cup K_1 \cup \cup \ldots \cup K_{n-1}$ has finitely many connected components. $\square$

Lemma 2 Let $f : T \longrightarrow T$ be a continuous map of a tree T. Then $x \in \Omega(f)$ if and only if for each $\epsilon > 0$ and each $L \in \mathbb{N}$, there is some $y \in T$ and some integer $m > L$ with $d(x, y) < \epsilon$ and $f^m(y) = x$. Equivalently, $x \in \Omega(f)$ if and only if there are $y_i \longrightarrow x$ and $n_i \longrightarrow \infty$ such that $f^{n_i}(y_i) = x$ for each $i \in \mathbb{N}$.

Proof. The sufficiency is easy and now we show the necessity. Let $x \in \Omega(f)$. Without loss of generality we assume that x is not a periodic point.

As $x \in \Omega(f)$, by the definition there are $y_i \longrightarrow x$ and $m_i \in \mathbb{N}$ such that $f^{m_i}(y_i) \longrightarrow x$. Since x is not periodic it is easy to check $m_i \longrightarrow \infty$. Assume the contrary that there is $\epsilon > 0$ and $L \in \mathbb{N}$ such that for any $y \in T$ with $d(x, y) < \epsilon$ and any $m > L$ we have $f^m(y) \neq x$. Let $U = \{y \in T : d(x, y) < \epsilon_1\}$, where $\epsilon_1 < \epsilon$ is small enough so that U is a connected neighborhood of x. Hence we have $x \notin K = f^{L+1}(U) \cup f^{L+2}(U) \cup \cdots$ and $x \notin f^j(K) = f^{L+j+1}(U) \cup f^{L+j+2}(U) \cup \cdots, j \in \mathbb{N}$. On the other hand, there is some $N > 0$ such that for any $i > N$, $y_i \in U$. For any fixed integer $j > 0$, choose $M > N$ such that $m_i > L + j$ for any $i > M$. Hence $\{f^{m_{M+1}-j}(y_{M+1}), f^{m_{M+2}-j}(y_{M+2}), \cdots\} \subset K$. Let $a_j \in \overline{K}$ be a limit point of this sequence. Then $x = f^j(a_j) \in f^j(\overline{K})$. So we have for any $j > 0$, $x \in f^j(\overline{K}) \setminus f^j(K)$.

As x is a non-wandering point, there is some $n > 0$ with $f^n(U) \cap U \neq \emptyset$. Hence $f^{n+L}(U) \cap f^L(U) \neq \emptyset$, and by Lemma 1 K has finitely many connected components. Since $a_j \in \overline{K} \setminus K$ and $\overline{K} \setminus K$ is a finite set, there are $j_2 > j_1 > 0$ such that $a_{j_1} = a_{j_2} = a$. So $f^{j_1}(a) = f^{j_2}(a) = x$. Thus $f^{j_2-j_1}(x) = f^{j_2-j_1}(f^{j_1}(a)) = f^{j_2}(a) = x$, i.e. x is a periodic point. This contradicts with our assumption. $\square$

Lemma 3 Let $f : T \longrightarrow T$ be a continuous map of a tree T, and S be a subtree of T. Then there is $y \in S$ such that either y is a fixed point of f or $y \in (V(T) \cap S) \cup E(S)$ such that $[y, f(y)] \cap S = \{y\}$. Clearly, if $x \in T$ such that $f(x), f(y)$ are belonging to the different connected components of $T \setminus S$, then $f([y,x]) \supset [y, f(x)]$.

The point y above is called a *p-fixed point (for S)*.

Proof. Let r_S be the retraction mapping. As $r_S \circ f|S : S \longrightarrow S$ is continuous and S has the fixed point property, there is a fixed point y of $r_S \circ f|S$. y is the point we need as T is uniquely arc-wise connected. □

Lemma 4 Let $a, c \in \mathbb{N}$ and $b \in \mathbb{Z}$. If $(a,c)|b$, then there exist $u, u' \in \mathbb{N}$ with $u, u' \leq \frac{c}{(c,a)}$ such that $c|(au+b)$ and $c|(-au'+b)$, where (x,y) is the greatest common divisor of integers x and y and $x|y$ means $\frac{y}{x} \in \mathbb{Z}$.

Proof. As $(\frac{c}{(a,c)}, \frac{a}{(a,c)}) = 1$ and $(a,c)|b$, there is $u \in \mathbb{N}$ with $u \leq \frac{c}{(c,a)}$ such that $\frac{c}{(a,c)}|(\frac{a}{(a,c)}u + \frac{b}{(a,c)})$. Thus $c|(au+b)$. By the same reasoning there is $u' \in \mathbb{N}$ with with $u' \leq \frac{c}{(c,a)}$ such that $c|(-au'+b)$. □

The next two lemmas are the results in [3].

Lemma 5 Let $f : T \longrightarrow T$ be a continuous map of a tree T. If there is $n \geq 2$ such that $x \in \Omega(f) \setminus \Omega(f^n)$, then $Orb(x,f)$ is finite.

Lemma 6 Let $f : T \longrightarrow T$ be a continuous map of a tree T and $p \geq 3$ be a prime number. Then $\Omega(f^{p^\lambda}) = \Omega(f^{p^{\lambda+1}})$ whenever $p^{\lambda+1} > End(T)$.

3 Examples

In this section we will give two examples which also serve as the necessities of Theorem 1 and Theorem 2.

Example 1. Let T be a star and $End(T) = pq$, where p, q are distinct prime numbers, then there is $f \in C(T,T)$ such that $\Omega(f^p) \cup \Omega(f^q) \neq \Omega(f)$.

Proof. Let T be a $pq-$star with $E(T) = \{e_1, e_2, \ldots, e_{pq}\}$ and y be the unique vertex with valence large than 1. Take $x, b, c \in (e_1, y)$ with $e_1 < x < b < c < y$ for some orientation of $[e_1, y]$. Construct a continuous $f : T \longrightarrow T$ such that

1) $f(e_1) = c$, $f(x) = y$, $f(b) = y$, $f(c) = e_2$, $f(e_i) = f(e_{i+1}), 2 \leq i \leq pq-1$, $f(e_{pq}) = x$, and $f(y) = y$ (see Figure 1).

2) f is piece-wise linear with respect to $x, b, c, y, e_1, \ldots, e_{pq}$.

If B is a neighborhood of x which is small enough, then we readily have that $f^{npq+1}(B) \cap B \neq \emptyset$ and $f^{npq+r}(B) \cap B = \emptyset$, $2 \leq r \leq pq$ for some $n \in \mathbb{N}$. That is, $x \in \Omega(f) \setminus (\Omega(f^p) \cup \Omega(f^q))$. □

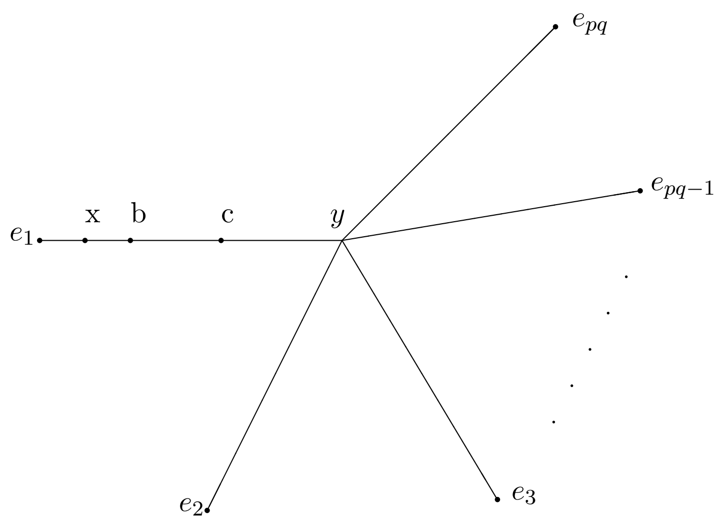

Figure 1

Example 2 Let T be a star and $End(T) = p + q + 1$, where p, q are distinct prime numbers, then there is $f \in C(T, T)$ such that $\Omega(f^p) \cap \Omega(f^q) \neq \Omega(f^{pq})$.

Proof. Assume T is a $(p + q + 1)-$star with $E(T) = \{e_1, e_2, \ldots, e_{p+q+1}\}$ and y is the unique vertex with valence large than 1. Suppose that $x, p_1, p_2, p_3, p_4, p_5$ and p_6 are the midpoints of $[e_1, y]$, $[e_2, y]$, $[e_{p+1}, y]$, $[p_2, y]$, $[e_{p+2}, y]$, $[e_{p+q+1}, y]$ and $[p_5, y]$ respectively. Moreover, we take $\{a_i\}, \{a'_i\}, \{b_i\}, \{b'_i\} \subset (e_1, x)$ such that for some orientation of $[e_1, x]$

$$e_1 < a'_1 < a_1 < b'_1 < b_1 < a'_2 < a_2 < b'_2 < b_2 < \ldots < x$$

and $\lim a_i = x$. Let $b_0 = e_1$.

We construct a continuous $f : T \longrightarrow T$ as follows (see Figure 2 and Figure 3).

1) $f(x) = y, f(p_1) = e_3,\ f(p_2) = y, f(p_3) = p_1,\ f(e_i) = f(e_{i+1}), 2 \leq i \leq p,\ f(e_{p+1}) = x, f(p_4) = e_{p+3},\ f(p_5) = y, f(p_6) = p_4,\ f(e_i) = f(e_{i+1}), p + 2 \leq i \leq p + q$, $f(e_{p+q+1}) = x$, and $f(y) = y$.

2) Set $f(a_i) = f(b_i) = y$, where $i \in \mathbb{N}$ and $j \in \{0\} \cup \mathbb{N}$. Let $f(a'_i) \in [y, e_2]$ and $f(b'_i) \in [y, e_{p+2}]$, $i \in \mathbb{N}$ such that $p_1 < f(a'_1) < f(a'_2) < \ldots < y$ and $p_4 < f(b'_1) < f(b'_2) < \ldots < y$ for some orientations of $[p_1, y]$ and $[p_4, y]$.

3) f is piece-wise linear with respect to the points mentioned above.

If B is a neighborhood of x which is small enough, for each $n \in \mathbb{N}$ we have:

(1)$f^{np+1}(B) \supset [x, p_1]$,

(2)$f^{np+r}(B) \supset [y, e_{r+1}],\ 2 \leq r \leq p$,

(3)$f^{nq+1}(B) \supset [x, p_4]$,

(4)$f^{nq+r}(B) \supset [y, e_{p+r+1}],\ 2 \leq r \leq q$.

By (1) and (2), we have $x \in \Omega(f^l)$ for each l which satisfies $(l, p) = 1$ and $x \notin \Omega(f^{pk})$ for each $k \in \mathbb{N}$. By (3) and (4), we have $x \in \Omega(f^l)$ for each l which satisfies $(l, q) = 1$ and $x \notin \Omega(f^{qk})$ for each $k \in \mathbb{N}$.

Especially, we have $x \in (\Omega(f^p) \cap \Omega(f^q)) \setminus \Omega(f^{pq})$. $\square$

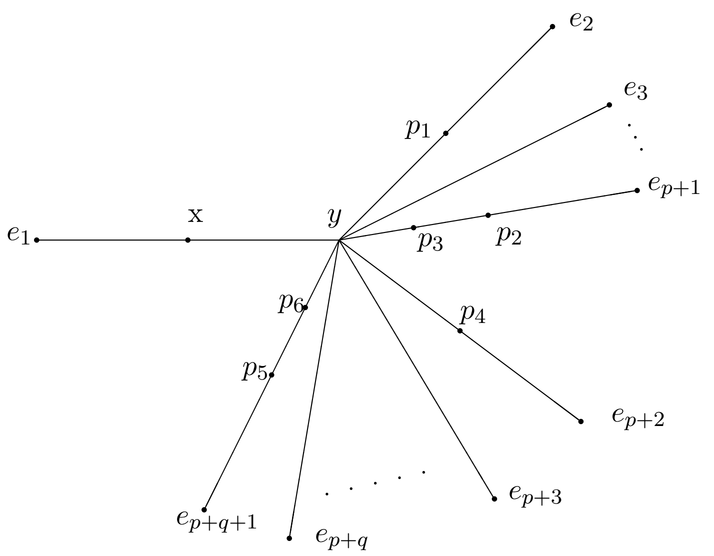

Figure 2

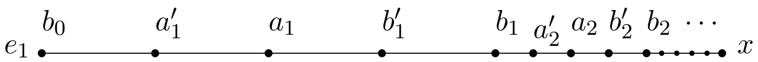

Figure 3

4 Proofs of the main results

Before we start to prove our theorems, we study some properties under the condition that $f \in C(T, T), x \in \Omega(f^p) \setminus \Omega(f^{pq})$, where T is a tree and p, q are natural numbers. We will introduce some notations which will be used in our proofs.

Let f be a continuous map of a tree T and $x \in \Omega(f^p) \setminus \Omega(f^{pq})$. By Lemma 2 there are $u_i \longrightarrow x$ and $n_i \longrightarrow \infty$ such that $f^{pn_i}(u_i) = x$ for each $i \in \mathbb{N}$ and there is a neighborhood B of x with $f^{pqk}(B) \cap B = \emptyset$ for each $k \in \mathbb{N}$. Let K be the smallest connected subset of T which contains $\cup_{i=0}^{\infty} f^i(B)$. Set $P = Orb(x, f) \cap P(f)$. Let y be a p-fixed point for $[P]$ and l_0 be the least integer with $y \in f^{l_0}(B)$.

Set $E(\overline{K}) = \{e_1, \ldots, e_l\}$ with $l = End(\overline{K})$, and we call $[y, e_i]$ a *segment*, where $1 \leq i \leq l$. A segment $[y, e_w]$ is of $u-$*type*, if for every natural number M there is some $j \in \mathbb{N}$ with $Card(\{f^i(u_j) | 1 \leq i \leq pn_j - 1\} \cap [y, e_w]) \geq M$.

Set $A_u(z) = \{f^{l_0+au+w}(z) | 0 \leq w \leq a-1\}$, where $a, u \in \mathbb{N}$ and $z \in B$. If there is some b with $1 \leq b \leq a$ such that for every $m \in \mathbb{N}$, there are j and $t_1, t_2, \ldots, t_m$ such that $Card(A_{t_k}(u_j) \cap [y, e_w]) \geq b$ where $k = 1, 2, \ldots, m$, then $[y, e_w]$ is called a *$u-$type segment which contains at least b points mod a*. Let $[y, e_w]$ be $u-$type segment which contains at least b points mod a. Without difficulty, by the above definition we have that there are fixed $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_b \leq a - 1$ such that for every $m \in \mathbb{N}$, there are j and $t_1, t_2, \ldots, t_m$ such that

$$\{f^{l_0+at_k+\alpha_i}(u_j)|0 \le i \le b\} \subset [y, e_w],\ k = 1, 2, \ldots, m.$$

In the sequel, we take this as the definition of $u-$type segment which contains at least b points mod a, and briefly we call $[y, e_w]$ a $(u; \alpha_1, \alpha_2, \ldots, \alpha_b; mod\ a)-type$ *segment*. It is easily seen that if $\frac{a}{l} > b$, then there exists at least one $u-$type segment which contains at least b points mod a.

Giving the notations, we will show the following propositions. The proof of Proposition 1 is similar to that of the Theorem 3.1 of [3]. For completeness, we include a proof.

Proposition 1 Let $[y, e_w]$ be $(u; \alpha_1, \alpha_2, \ldots, \alpha_b; \text{mod } pqn)-$type segment, where n is a fixed natural number and $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_b \le pqn - 1$, then $(\alpha_j - \alpha_i, pq) \not| p$ for each $1 \le i < j \le b$.

In particular, if $(p, q) = 1$ and q is a prime number, then $q|(\alpha_j - \alpha_i)$ for each $1 \le i < j \le b$.

Proof. Assume the contrary. That is, there are $1 \le i_0 \le j_0 \le b$ such that $(\alpha_{j_0} - \alpha_{i_0}, pq)|p$. Set $a = \alpha_{j_0} - \alpha_{i_0}$, then $(a, pq)|pn_j$. By Lemma 4 there are natural numbers $u, u' \le \frac{pq}{(pq,a)}$ with $pq|(au + pn_j)$ and $pq|(-au' + pn_j)$.

Take $m = pq + 1$ in the above definition, and let j and $t_1 < t_2 < \ldots < t_m$ be the corresponding natural numbers. Let $a_i = l_0 + t_i pqn + \alpha_{i_0}$ and $b_i = l_0 + t_i pqn + \alpha_{j_0}$.

We claim:

$$y < f^{b_m}(u_j) < f^{a_m}(u_j) < \ldots < f^{b_2}(u_j) < f^{a_2}(u_j) < f^{b_1}(u_j) < f^{a_1}(u_j),$$

if we define an orientation of $[y, e_w]$ such that $y < e_w$.

Proof of the claim Let $c_{2t-1} = a_t$ and $c_{2t} = b_t$, $1 \le t \le m$. Then $c_1 < c_2 < \ldots < c_{2m}$. Assume that there are $1 \le l_1 < l_2 \le 2m$ with $(l_2 - l_1, 2) = 1$ such that $f^{c_{l_1}}(u_j) \in [y, f^{c_{l_2}}(u_j)]$. Then

$$f^{c_{l_2}-c_{l_1}}([y, f^{c_{l_2}}(u_j)] \supset f^{c_{l_2}-c_{l_1}}([y, f^{c_{l_1}}(u_j)]) \supset [y, f^{c_{l_2}}(u_j)].$$

Hence $f^{b(c_{l_2}-c_{l_1})}([y, f^{c_{l_2}}(u_j)] \supset [y, f^{c_{l_2}}(u_j)]$, with $b = u$ or u'. As

$$\begin{aligned} x &\in f^{pn_j-c_{l_2}}([y, f^{c_{l_2}}(u_j)]) \subset f^{pn_j-c_{l_2}+b(c_{l_2}-c_{l_1})}([y, f^{c_{l_2}}(u_j)]) \\ &\subset f^{pn_j-c_{l_2}+b(c_{l_2}-c_{l_1})}(f^{c_{l_2}}(B)) = f^{pn_j+b(c_{l_2}-c_{l_1})}(B) \end{aligned}$$

and $pq|(pn_j + b(c_{l_2} - c_{l_1}))$, a contradiction.

Hence we have

$$y < f^{b_m}(u_j) < f^{a_m}(u_j) < \ldots < f^{b_2}(u_j) < f^{a_2}(u_j) < f^{b_1}(u_j) < f^{a_1}(u_j).$$

This ends the proof of the claim.

Hence by the claim we have

$$f^{b_s-a_s}([y, f^{a_s}(u_j)]) \supset [y, f^{b_s}(u_j)] \supset [y, f^{a_{s+1}}(u_j)]$$

for each $1 \le s \le pq$. Thus

$$x \in [x, y] \subset f^{pn_j-a_{u+1}}([y, f^{a_{u+1}}(u_j)]) \subset f^{pn_j-a_{u+1}+a_1+(b_1-a_1)+\ldots+(b_u-a_u)}(B).$$

As

$$pq|(pn_j - a_{u+1} + a_1 + (b_1 - a_1) + \ldots + (b_u - a_u)),$$

a contradiction. □

By Proposition 1, we can get Proposition 2 readily.

Proposition 2 If $(p, q) = 1$ and q is a prime number, then each $u-$type segment contains at most p points mod pq. So the number of $u-$type segment of K is not less than q.

Now we are ready to prove our theorems.

Theorem 1 Let T be a star and p, q be two distinct prime numbers, then $\Omega(f^p) \cup \Omega(f^q) = \Omega(f)$ for each $f \in C(T, T)$ if and only if $pq > End(T)$.

Proof. To show the necessity, it is enough to show that whenever $End(T) \geq pq$, there is some f such that $\Omega(f^p) \cup \Omega(f^q) \neq \Omega(f)$. This is done in Example 1 of the last section.

Now suppose $pq > End(T)$ and $\Omega(f^p) \cup \Omega(f^q) \neq \Omega(f)$. Then there is $x \in \Omega(f) \setminus (\Omega(f^p) \cup \Omega(f^q))$. By Lemma 2, there are $v_i \longrightarrow x$ and $n_i \longrightarrow \infty$ such that$f^{n_i}(v_i) = x$ for each $i \in \mathbb{N}$ and there is an open connected neighborhood B of x with $f^{pk}(B) \cap B = \emptyset$ and $f^{qk}(B) \cap B = \emptyset$ for each $k \in \mathbb{N}$. We may assume that $v_i \in B$ for each $i \in \mathbb{N}$. Set $K = \cup_{i=0}^{\infty} f^i(B)$ and $E(\overline{K}) = \{e_1, \ldots, e_l\}$ with $l = End(\overline{K})$.

First suppose K is connected. Set $P = Orb(x, f) \cap P(f)$, and let y be a p-fixed point for $[P]$. As $pq > End(T)$, there is a $v-$type segment $[y, e_w]$ which contains at least 2 points mod pq. As $x \in \Omega(f) \setminus \Omega(f^p)$ and $x \in \Omega \setminus \Omega(f^q)$, $[y, e_w]$ is their common $v-$type segment. Assume that $[y, e_w]$ is of $(v; \alpha, \beta; \bmod pq)-$type.

By Proposition 1, $(\beta - \alpha, p) = p$ and $(\beta - \alpha, q) = q$, $1 \leq \alpha < \beta \leq pq - 1$. So $pq|(\beta - \alpha)$. As $1 \leq \alpha - \beta \leq pq - 1$ and p,q are distinct prime numbers, $pq \not| (\beta - \alpha)$, a contradiction.

Now suppose K is not connected, then by Lemma 1 K has finitely many connected components $K_1, \ldots, K_r$ with $f(K_1) \subset K_2, \ \ldots, f(K_r) \subset K_1$.

Let $g = f^r$ and assume $x \in K_1$. Then $g(K_1) \subset K_1$ and $x \in \Omega(g) \setminus (\Omega(g^p) \cup \Omega(g^q))$. As $pq > \mathrm{End}(T) \geq End(\overline{K_1})$ and K_1 is connected, a contradiction arrives again if we replace f by f^r and use what we just proved.

Hence we get $\Omega(f) \setminus (\Omega(f^p) \cup \Omega(f^q)) = \emptyset$, i.e. $\Omega(f^p) \cup \Omega(f^q) = \Omega(f)$. □

The same method can be used to show

Theorem 1' Let T be a star and $p_1, p_2, \ldots, p_k$ be distinct prime numbers, then $\Omega(f^{p_1}) \cup \Omega(f^{p_2}) \cup \ldots \cup \Omega(f^{p_k}) = \Omega(f)$ for each $f \in C(T, T)$ if and only if $p_1 p_2 \ldots p_k > End(T)$.

Theorem 2 Let T be a star and p, q be two distinct prime numbers, then $\Omega(f^p) \cap \Omega(f^q) = \Omega(f^{pq})$ for each $f \in C(T, T)$ if and only if $p + q \geq End(T)$.

Proof. To show the necessity, it is enough to show that whenever $\mathrm{End}(T) \geq p+q+1$, there is some f such that $\Omega(f^p) \cap \Omega(f^q) \neq \Omega(f^{pq})$. This is done in Example 2 of the last section.

Now suppose $p + q \geq \mathrm{End}(T)$ and $\Omega(f^p) \cap \Omega(f^q) \neq \Omega(f^{pq})$. Then there is $x \in (\Omega(f^p) \cap \Omega(f^q)) \setminus \Omega(f^{pq})$. By Lemma 2, there are $u_i \longrightarrow x$ and $n_i \longrightarrow \infty$ such that $f^{pn_i}(u_i) = x$, and $v_i \longrightarrow x$ and $m_i \longrightarrow \infty$ such that $f^{qm_i}(v_i) = x$ for each $i \in \mathbb{N}$. Moreover, there is an open connected neighborhood B of x with $f^{pqk}(B) \cap B = \emptyset$ for each $k \in \mathbb{N}$. We may assume that $u_i, v_i \in B$ for each $i \in \mathbb{N}$. Set $K = \cup_{i=0}^{\infty} f^i(B)$ and $E(\overline{K}) = \{e_1, \dots, e_l\}$ with $l = \text{End}(\overline{K})$.

First suppose K is connected. We divide the proof into several steps.

Step 1. Set $P = Orb(x, f) \cap P(f)$ and let y be a p-fixed point for $[P]$. For each fixed natural number N, $[x, y] \not\subset f^N([x, y])$.
Proof of Step 1: Let $l_0 \in \mathbb{N}$ such that $y \in f^{l_0}(B)$, then there are n_k, m_k with $pn_k > l_0$ and $qm_k > l_0$ as $n_i \longrightarrow \infty$ and $m_i \longrightarrow \infty$. Hence $f^{pn_k}(B) \supset [f^{pn_k}(u_k), y] = [x, y]$. Similarly we have $f^{qm_k}(B) \supset [x, y]$.

Assume the contrary. That is, there is N such that $[x, y] \subset f^N([x, y])$. As $[x, y] \subset f^{kN}([x, y])$ for each $k \in \mathbb{N}$, we may assume $N > pq + l_0$. Since p, q are distinct prime numbers, we have either $(p, N) = 1$ or $(p, N) = 1$ or $(pq, N) = pq$. We discuss them respectively.

Case a: $(p, N) = 1$. As $(p, q) = 1$, we have $(p, qN) = 1$. By Lemma 4, there is $t \in \mathbb{N}$ such that $p|(qm_k + qNt)$. Hence $x \in [x, y] \subset f^{Nqt}([x, y]) \subset f^{qm_k+qNt}(B)$. As $pq|(qm_k + qNt)$, a contradiction.

Case b: $(q, N) = 1$. The proof is similar to Case a.

Case c: $pq|N$. As $[x, y] \subset f^N([x, y])$, there is $z \in (x, y)$ such that $f^N(z) = x$. As $pq > p + q \geq \text{End}(T) \geq l$, there are $l_0 \leq i \neq j \leq l_0 + pq - 1$ such that $f^i(z)$ and $f^j(z)$ are in the same segment $[y, e_w]$. We may assume $f^i(z) \in [y, f^j(z)]$. Then we have

$$[x, y] \subset f^{N-i}([y, f^i(z)]) \subset f^{N-i}([y, f^j(z)]) \subset f^{N-i+j}([y, z]) \subset f^{N-i+j}([x, y]).$$

Let $t = N + j - i$. As $pq \not| t$, we have either $(t, p) = 1$ or $(t, q) = 1$. Thus, we get contradiction as in Case a or Case b. This ends proof of Step 1.

Step 2. For each $u_j,\ v_k$, we have $f^m(u_j),\ f^n(v_k) \notin (x, y)$ and $f^m(u_j),\ f^n(v_k) \notin [P] \setminus P$ for each $m,\ n \in \mathbb{N}$.
Proof of Step 2: Assume there are u_j and $n \in \mathbb{N}$ such that $f^n(u_j) \in [x, y], 1 \leq n \leq pn_j - 1$, then we have $f^N([x, y]) = f^{pn_j-n}([x, y]) \supset f^{pn_j-n}([f^n(u_j), y]) \supset [x, y]$, where $N = pn_j - n$. This contradicts with Step 1.

Assume there are u_j and $n \in \mathbb{N}$ such that $f^n(u_j) \in [P] \setminus P$, then there is $z \in P$ such that $f^n(u_j) \in [y, z]$. Set $z = f^a(x), a \in \mathbb{N}$, then we have $[y, z] \subset f^a([x, y])$. Hence

$$[x, y] \subset f^{pn_j-n}([f^n(u_j), y]) \subset f^{pn_j-n}([z, y]) \subset f^{pn_j-n+a}([x, y]) = f^N([x, y]),$$

where $N = pn_j - n + a$. This contradicts with the Step 1 again.

For v_k, we can prove in the same way. This ends the proof of Step 2.

Step 3. There is a segment $[y, e_w]$ that is not only a $u-$type segment which contains at least b_1 points mod pq but also a $v-$type segment which contains at least b_2 points mod pq, where $b_1, b_2 \in \mathbb{N}$ and $b_1 + b_2 \geq 3$.
Proof of Step 3: By Proposition 2 the number of $u-$type segments is not less than q and the number of $v-$type segments is not less than p. As $p + q \geq End(\overline{K})$, there are segments which are not only $u-$type but also $v-$type. Set the number of such segments be s. We assume that each segment which is not only $u-$type but also $v-$type only contains one point mod pq. Set the number of $u-$type ($v-$type) but not $v-$type (resp. $u-$type) segments be s_1 (resp. s_2), then $s_1 + s_2 + s \leq End(\overline{K}) - 1$ by Step 2.

We claim either $\frac{(pq-s)}{s_1} > p$ or $\frac{(pq-s)}{s_2} > q$. Assume the contrary. That is, $\frac{(pq-s)}{s_1} \le p$ and $\frac{(pq-s)}{s_2} \le q$. Hence $q - \frac{s}{p} \le s_1, p - \frac{s}{q} \le s_2$. Hence

$$p + q - \frac{s}{p} - \frac{s}{q} \le s_1 + s_2 \le End(\overline{K}) - s - 1 < p + q - s.$$

Hence we have $pq < p + q$, a contradiction.

So we have either $\frac{(pq-s)}{s_1} > p$ or $\frac{(pq-s)}{s_2} > q$. It contradicts with Proposition 2. This ends the proof of Step 3.

Step 4. Now we will give a contradiction.

Let $[y, e_w]$ be the segment in Step 3 and $b_1 \ge 2, b_2 \ge 1$. Assume $[y, e_w]$ is $(u; \alpha, \beta; \bmod pq)$ and $(v; \gamma; \bmod pq)-$type and we define an orientation of $[y, e_w]$ such that $y < e_w$.

We claim that we can choose $k_1, k_2, \ldots, k_h; j_1, j_2, \ldots, j_h$, and $t'_{k_1} < t'_{k_2} < \ldots < t'_{k_h}, t_{j_1} < t_{j_2} < \ldots < t_{j_h}$, where $h = pq + 1$ such that

$$y < f^{b_h}(v_{k_h}) < f^{b_{h-1}}(v_{k_{h-1}}) < \ldots < f^{b_1}(v_{k_1}) < e_w$$

and $f^{a_s}(u_{j_s}), f^{a'_s}(u_{j_s}) \in [f^{b_{s+1}}(v_{k_{s+1}}), f^{b_s}(v_{k_s})]$, where $a_s = l_0 + pqt_{j_s} + \alpha, a'_s = l_0 + pqt_{j_s} + \beta$ and $b_s = l_0 + pqt'_{k_s} + \gamma$. To show the claim, first we choose k_1 arbitrarily and then we choose j_1 such that $f^{a_1}(u_{j_1}), f^{a'_1}(u_{j_1})$ are on the left of $f^{b_1}(v_{k_1})$. We then choose k_2 such that $f^{b_2}(v_{k_2})$ is on the left of the three points above and j_2 such that $f^{a_2}(u_{j_2})$, $f^{a'_2}(u_{j_2})$ are on the left of $f^{b_2}(v_{k_2})$. Repeating the above argument, we can get what we have claimed (the reason why we can do in such way depends on the results of Step 1 and Step 2).

By Proposition 1 $(\beta - \alpha, p) = 1$. Let $a = \beta - \alpha$, then $(a, pq) | qm_{k_h}$. By Lemma 4 there is $t \in \{1, 2, \ldots, pq\}$ such that $pq | (qm_{k_h} + at)$. As

$$\begin{aligned} [y, f^{b_s}(v_{k_s})] &\subset [y, f^{a'_{s-1}}(u_{j_{s-1}})] \subset f^{a'_{s-1} - a_{s-1}}([y, f^{a_{s-1}}(u_{j_{s-1}})]) \\ &\subset f^{a'_{s-1} - a_{s-1}}([y, f^{b_{s-1}}(v_{k_{s-1}})]) \end{aligned}$$

for each $2 \le s \le h$, we have

$$\begin{aligned} [x, y] &\subset f^{qm_{k_h} - b_h}([y, f^{b_h}(v_{k_h})]) \\ &\subset f^{qm_{k_h} - b_h + (a'_{h-1} - a_{h-1})}([y, f^{b_{h-1}}(v_{k_{h-1}})]) \\ &\subset \ldots \\ &\subset f^{qm_{k_h} - b_h + (a'_{h-1} - a_{h-1}) + \ldots + (a'_{h-t} - a_{h-t})}([y, f^{b_{h-t}}(v_{h-t})]) \\ &\subset f^{qm_{k_h} - b_h + b_{h-t} + (a'_{h-1} - a_{h-1}) + \ldots + (a'_{h-t} - a_{h-t})}(B). \end{aligned}$$

As $pq | (qm_{k_h} - b_h + b_{h-t} + (a'_{h-1} - a_{h-1}) + \ldots + (a'_{h-t} - a_{h-t}))$, a contradiction.

Now suppose K is not connected, then by Lemma 1 K has finitely many connected components $K_1, \ldots, K_r$ with $f(K_1) \subset K_2, \ \ldots, f(K_r) \subset K_1$.

Let $g = f^r|_{K_1}$ and assume $x \in K_1$. It is easy to see $x \in (\Omega(g^p) \cap \Omega(g^q)) \setminus \Omega(g^{pq})$. As $p + q \ge End(T) \ge End(\overline{K_1})$, we can replace f by g and use what we just proved.

To sum up, we have proved $\Omega(f^p)\cap\Omega(f^q)=\Omega(f^{pq})$. $\Box$

Theorem 2' Let T be a star and $p_1<p_2<\ldots<p_k$ be distinct prime numbers, then $\Omega(f^{p_1})\cap\Omega(f^{p_2})\cap\ldots\cap\Omega(f^{p_k})=\Omega(f^{p_1p_2\cdots p_k})$ for each $f\in C(T,T)$ if and only if $p_1+p_2\geq End(T)$.

Proof. To show the necessity, it is enough to show that whenever $End(T)\geq p_1+p_2+1$, there is some f such that $\Omega(f^{p_1})\cap\Omega(f^{p_2})\cap\ldots\cap\Omega(f^{p_k})\neq\Omega(f^{p_1p_2\cdots p_k})$. This is done in Example 2 of the last section.

Now suppose $p_1+p_2\geq End(T)$. Then $p_1+p_i\geq End(T), 2\leq i\leq k$, and by Theorem 2 we have $\Omega(f^{p_1})\cap\Omega(f^{p_2})\cap\ldots\cap\Omega(f^{p_k})=\Omega(f^{p_1p_2})\cap\Omega(f^{p_1p_3})\cap\ldots\cap\Omega(f^{p_1p_k})$. As $p_2+p_i\geq End(T), 3\leq i\leq k$, and apply Theorem 2 to f^{p_1} we have $\Omega(f^{p_1p_2})\cap\Omega(f^{p_1p_3})\cap\ldots\cap\Omega(f^{p_1p_k})=\Omega(f^{p_1p_2p_3})\cap\Omega(f^{p_1p_2p_4})\cap\ldots\cap\Omega(f^{p_1p_2p_k})$. Then we apply Theorem 2 to $f^{p_1p_2}$. Inductively, after finite steps we have $\Omega(f^{p_1})\cap\Omega(f^{p_2})\cap\ldots\cap\Omega(f^{p_k})=\Omega(f^{p_1p_2\cdots p_k})$. $\Box$

Proof of Theorem 3: Let T be a 3-star or 4-star and $f\in C(T,T)$. By the previous lemmas and theorems we have:

(1) $\Omega(f)\supseteq\Omega(f^2)\supseteq\Omega(f^{2^2})\supseteq\ldots,$

(2) $\Omega(f)\supseteq\Omega(f^3)=\Omega(f^{3^3})=\ldots,$ (Lemma 6)

(3) $\Omega(f^{p^\lambda})=\Omega(f^{p^{\lambda+1}})$, where $\lambda\geq 0$ and $p\geq 5$ is a prime number, (Lemma 6)

(4) $\Omega(f^2)\cup\Omega(f^3)=\Omega(f)$, (Theorem 1)

(5) $\Omega(f^2)\cap\Omega(f^3)=\Omega(f^6)$. (Theorem 2)

(1.1) For each $n\in\mathbb{N}$ let $n=2^k3^tm$, where $k,t\in\mathbb{Z}_+, m\in\mathbb{N}$ and $(m,6)=1$, then by (1), (2) and (3) we have

$$\Omega(f^n)=\Omega(f^{2^k3^tm})=\begin{cases}\Omega(f^{2^k}), & \text{if } t=0\\ \Omega(f^{2^k3}). & \text{if } t>0\end{cases}$$

(1.2) For any continuous map f of T, by (4) and (5) we have

$$\begin{aligned}\Omega(f)\setminus\Omega(f^2) &= \Omega(f^3)\setminus\Omega(f^6)\\ \Omega(f)\setminus\Omega(f^3) &= \Omega(f^2)\setminus\Omega(f^6)\end{aligned}$$

Replacing f by f^{2^i}, $i\in\mathbb{N}$ we get (a) and (b) respectively.

(2) We now show in the graph A any pre-assigned sequence of equalities and strict containments in the first line and the first column can be realized.

If $\Omega(f)=\Omega(f^3)$, then any pre-assigned sequence of equalities and strict containments in the first line can be realized for an interval map [2], and obviously can be realized for a 3−star map.

If $\Omega(f)\neq\Omega(f^3)$, we construct f as follows: Let T be a 3-star with $E(T)=\{e_1,e_2,e_3\}$ and let y be the unique vertex with valence larger than 1. Take $a,b\in(e_1,y)$ such that $a\in(e_1,b)$. Set $T_1=[b,e_2,e_3]$ and $T_2=[e_1,a]$. By [3] we can construct $f_1\in C(T_1,T_1)$ such that $\Omega(f_1)\neq\Omega(f_1^3)$. By [2] we can construct $f_2\in C(T_2,T_2)$ such that any pre-assigned sequence of equalities and strict containments in the first line can be realized by f_2. Now define $f : T \longrightarrow T$ such that $f|_{T_1} = f_1,\ f|_{T_2} = f_2$ and f is linear in $[a, b]$. Then f is the continuous map we need.

It is easy to obtain maps of 4-star by a small modification. □

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E-mail: songshao@ustc.edu.cn, yexd@ustc.edu.cn