NIL BOHR$_0$-SETS, POINCARÉ RECURRENCE AND GENERALIZED POLYNOMIALS

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Abstract. The problem which can be viewed as the higher order version of an old question concerning Bohr sets is investigated: for any $d \in \mathbb{N}$ does the collection of $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$ with S syndetic coincide with that of Nil_d Bohr$_0$-sets?

In this paper it is proved that Nil_d Bohr$_0$-sets could be characterized via generalized polynomials, and applying this result one side of the problem could be answered affirmatively: for any Nil_d Bohr$_0$-set A, there exists a syndetic set S such that $A \supset \{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$. Note that other side of the problem can be deduced from some result by Bergelson-Host-Kra if modulo a set with zero density. As applications it is shown that the two collections coincide dynamically, i.e. both of them can be used to characterize higher order almost automorphic points.

1. Introduction

Combinatorial number theory attracts a lot of attention. In such a theory, problems concerning Bohr sets are extensively studied, and have a long history which at least could be traced back to the work of Veech in 1968 [29]. Bohr sets are fundamentally abelian in nature. Nowadays it has become apparent that a higher order non-abelian Fourier analysis plays a role both in combinatorial number theory and ergodic theory. Related to this, a higher-order version of Bohr sets, namely Nil_d Bohr$_0$-sets, was introduced in [18]. For the recent results obtained by Katznelson, Bergelson-Furstenberg-Weiss and Host-Kra see [23, 3, 18].

1.1. **Nil-Bohr sets.** There are several equivalent definitions for Bohr sets. Here is the one easy to understand: a subset $A \subseteq \mathbb{Z}$ is a *Bohr set* if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T}^m$, and an open set $U \subseteq \mathbb{T}^m$ such that $\{n \in \mathbb{Z} : n\alpha \in U\}$ is contained in A; the set A is a *Bohr$_0$-set* if additionally $0 \in U$.

It is not hard to see that if (X, T) is a minimal equicontinuous system, $x \in X$ and U is a neighborhood of x, then $N(x, U) =: \{n \in \mathbb{Z} : T^n x \in U\}$ contains $S - S =: \{a - b : a, b \in S\}$ with S syndetic, i.e. with a bounded gap. An old question concerning Bohr sets is

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Problem A-I: Let S be a syndetic set of $\mathbb{Z}$, is $S-S$ a Bohr$_0$-set?

That is, are the common differences of arithmetic progressions with length 2 appeared in a syndetic set a Bohr$_0$ set? Veech showed that it is at least "almost" true [29]. That is, given a syndetic set $S \subseteq \mathbb{Z}$, there is some subset N with density zero such that $(S-S)\Delta N$ is a Bohr$_0$-set.

A subset $A \subseteq \mathbb{Z}$ is a *Nil$_d$ Bohr$_0$-set* if there exist a d-step nilsystem (X,T), $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that $N(x_0,U) =: \{n \in \mathbb{Z} : T^n x_0 \in U\}$ is contained in A. Denote by $\mathcal{F}_{d,0}$ the family[1] consisting of all Nil$_d$ Bohr$_0$-sets. We can now formulate a higher order form of Problem A-I. We note that $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$ can be viewed as the common differences of arithmetic progressions with length $d+1$ appeared in the subset S. In fact, $S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset$ if and only if there is $m \in S$ with $m, m+n, \ldots, m+dn \in S$.

Problem B-I: [Higher order form of Problem A-I] Let $d \in \mathbb{N}$.

(1) *For any Nil$_d$ Bohr$_0$-set A, is it true that there is a syndetic subset S of $\mathbb{Z}$ with $A \supset \{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$?*
(2) *For any syndetic set S, is $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$ a Nil$_d$ Bohr$_0$-set?*

1.2. **Dynamical version of the higher order Bohr problem.** Sometimes combinatorial questions can be translated into dynamical ones by the Furstenberg correspondence principle, see Section 2.4. Using this principle, it can be shown that Problem A-I is equivalent to the following version:

Problem A-II: *For any minimal system (X,T) and any nonempty open set $U \subset X$, is the set $\{n \in \mathbb{Z} : U \cap T^{-n}U \neq \emptyset\}$ a Bohr$_0$-set?*

Similarly, Problem B-I has its dynamical version:

Problem B-II: [Dynamical version of Problem B-I] Let $d \in \mathbb{N}$.

(1) *For any Nil$_d$ Bohr$_0$-set A, it is true that there are a minimal system (X,T) and a non-empty open subset U of X with*
$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}?$$
(2) *For any minimal system (X,T) and any open non-empty $U \subset X$, is it true that $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$ a Nil$_d$ Bohr$_0$-set?*

It follows from some result by Bergelson-Host-Kra in [4] that Problem B-II(2) has a positive answer if ignoring a set with zero density. In fact, the authors [4] showed: Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$, then for all $A \in \mathcal{X}$ with $\mu(A) > 0$ the set $I = \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \ldots \cap T^{-dn}A) > 0\}$ is almost a Nil$_d$ Bohr$_0$-set, i.e. there is some subset N with density zero such that $I\Delta N$ is a Nil$_d$ Bohr$_0$-set.

[1]A collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ (or $\mathbb{N}$) is *a family* if it is hereditary upward, i.e. $F_1 \subseteq F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Any nonempty collection $\mathcal{A}$ of subsets of $\mathbb{Z}$ generates a family $\mathcal{F}(\mathcal{A}) := \{F \subseteq \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}$.

1.3. **Main results.** We will show that Problem B-II(1) has an affirmative answer. Namely, we will show

Theorem A: *Let $d \in \mathbb{N}$. If $A \subseteq \mathbb{Z}$ is a Nil$_d$ Bohr$_0$-set, then there exist a minimal d-step nilsystem (X,T) and a nonempty open set U of X with*

$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}.$$

As we said before for $d = 1$ Theorem A can be easily proved. To show Theorem A in the general case, we need to investigate the properties of $\mathcal{F}_{d,0}$. It is interesting that in the process to do this, generalized polynomials (see §4 for a definition) appear naturally. Generalized polynomials have been studied extensively, see for example the nice paper by Bergelson and Leibman [5] and references therein. After finishing this paper we even find that it also plays an important role in the recent work by Green, Tao and Ziegler [16]. In fact the special generalized polynomials defined in this paper are closely related to the nilcharacters defined there.

Let $\mathcal{F}_{GP_d}$ (resp. $\mathcal{F}_{SGP_d}$) be the family generated by the sets of forms

$$\bigcap_{i=1}^{k} \{n \in \mathbb{Z} : P_i(n) (\text{mod } \mathbb{Z}) \in (-\epsilon_i, \epsilon_i)\},$$

where $k \in \mathbb{N}$, $P_1, \ldots, P_k$ are generalized polynomials of degree $\leq d$ (resp. special generalized polynomials), and $\epsilon_i > 0$. For the precise definitions see §4.

The following theorem illustrates the relation between Nil$_d$ Bohr$_0$-sets and the sets defined above using generalized polynomials.

Theorem B: *Let $d \in \mathbb{N}$. Then $\mathcal{F}_{d,0} = \mathcal{F}_{GP_d}$.*

To prove Theorem B we first figure out a subclass of generalized polynomials (called special generalized polynomials) and show that $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$. When $d = 1$, we have $\mathcal{F}_{1,0} = \mathcal{F}_{SGP_1}$. This is the result of Katznelson [23], since $\mathcal{F}_{SGP_1}$ is generated by sets of forms $\cap_{i=1}^{k}\{n \in \mathbb{Z} : na_i (\text{mod } \mathbb{Z}) \in (-\epsilon_i, \epsilon_i)\}$ with $k \in \mathbb{N}$, $a_i \in \mathbb{R}$ and $\epsilon_i > 0$.

Theorem A follows from Theorem B and the following result:

Theorem C: *Let $d \in \mathbb{N}$. If $A \in \mathcal{F}_{GP_d}$, then there exist a minimal d-step nilsystem (X,T) and a nonempty open set U such that*

$$A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}.$$

The proof of Theorem B is divided into two parts, namely

Theorem B(1): $\mathcal{F}_{d,0} \subset \mathcal{F}_{GP_d}$ and

Theorem B(2): $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$.

The proof of Theorem B(1) is a theoretical argument using nilpotent Lie group theory; and the proofs of Theorem B(2) and Theorem C are very complicated construction and computation where nilpotent matrix Lie group is used.

Remark 1.1. Our definition of generalized polynomials is slight different from the ones defined in [5]. In fact we need to specialize the degree of the generalized polynomials which is not needed in [5]. Moreover, our Theorem B can be compared with Theorem A of Bergelson and Leibman proved in [5].

In [13, 12] Furstenberg introduced the notion of Poincaré recurrence sets and Birkhoff recurrence sets. Here is a generalization of the above notion. Let $d \in \mathbb{N}$. We say that $S \subset \mathbb{Z}$ is a set of *d-recurrence* if for every measure preserving dynamical system $(X, \mathcal{X}, \mu, T)$ and for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists $n \in S$ such that

$$\mu(A \cap T^{-n}A \cap \ldots \cap T^{-dn}A) > 0.$$

We say that $S \subseteq \mathbb{Z}$ is a set of *d-topological recurrence* if for every minimal system (X, T) and for every nonempty open subset U of X, there exists $n \in S$ such that

$$U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset.$$

Remark 1.2. The above definitions are slightly different from the ones introduced in [11], namely we do not require $n \neq 0$. The main reason we define in this way is that for each $A \in \mathcal{F}_{d,0}$, $0 \in A$. Thus $\{0\} \cup C \in \mathcal{F}_{d,0}^*$ for each $C \subset \mathbb{Z}$.

Let $\mathcal{F}_{Poi_d}$ (resp. $\mathcal{F}_{Bir_d}$) be the family generated by the collection of all sets of d-recurrence (resp. sets of d-topological recurrence). It is obvious by the above definition that $\mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d}$. Moreover, it is known that for each $d \in \mathbb{N}$, $\mathcal{F}_{Poi_d} \supsetneqq \mathcal{F}_{Poi_{d+1}}$ and $\mathcal{F}_{Bir_d} \supsetneqq \mathcal{F}_{Bir_{d+1}}$ [11]. Now we state a problem which is related to Problem B-II.

Problem B-III: *Is it true that* $\mathcal{F}_{Bir_d} = \mathcal{F}_{d,0}^*$*?*

where $\mathcal{F}_{d,0}^*$ is the dual family of $\mathcal{F}_{d,0}$, i.e. the collection of sets intersecting every Nil$_d$ Bohr$_0$ set.

An immediate corollary of Theorem A is:

Corollary D: *Let* $d \in \mathbb{N}$. *Then*

$$\mathcal{F}_{Poi_d} \subset \mathcal{F}_{Bir_d} \subset \mathcal{F}_{d,0}^*.$$

Note that $\mathcal{F}_{Poi_1} \neq \mathcal{F}_{Bir_1}$ [24]. Though we can not prove $\mathcal{F}_{Bir_d} = \mathcal{F}_{d,0}^*$, we will show that the two collections coincide "dynamically", i.e. both of them can be used to characterize higher order almost automorphic points, see §8.

1.4. **Organization of the paper.** We organize the paper as follows: In Section 2, we give some basic definitions, and particularly we show the equivalence of the Problems I, II and III. In Section 3 we recall basic facts related to nilpotent Lie groups and nilmanifolds, and study the properties of the metric on nilpotent matrix Lie groups. In Section 4, we introduce the notions related to generalized polynomials and special generalized polynomials, and give the basic properties. In the next three sections we show the main results. And in the last section, we state the applications of our main results, which will appear in a forthcoming article [22] by the same authors.

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2. Preliminaries

In this section we introduce some basic notions related to dynamical systems,explain how Bergelson-Host-Kra's result is related to Problem B-II and show the equivalence of the Problems I, II and III.

2.1. **Measurable and topological dynamics.** A (measurable) *system* is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is an invertible measure preserving transformation.

A *topological dynamical system*, referred to more succinctly as just a *system*, is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. We use $\rho(\cdot, \cdot)$ to denote the metric on X.

2.2. **Families and filters.** Since many statements of the paper are better stated using the notion of a family, we now give the definition. See [1] for more details.

2.2.1. *Furstenberg families.* We say that a collection $\mathcal{F}$ of subsets of $\mathbb{Z}$ is *a family* if it is hereditary upward, i.e. $F_1 \subseteq F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called *proper* if it is neither empty nor the entire power set of $\mathbb{Z}$, or, equivalently if $\mathbb{Z} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Any nonempty collection $\mathcal{A}$ of subsets of $\mathbb{Z}$ generates a family $\mathcal{F}(\mathcal{A}) := \{F \subseteq \mathbb{Z} : F \supset A \text{ for some } A \in \mathcal{A}\}$.

For a family $\mathcal{F}$ its *dual* is the family $\mathcal{F}^* := \{F \subseteq \mathbb{Z} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}$. It is not hard to see that $\mathcal{F}^* = \{F \subset \mathbb{Z} : \mathbb{Z} \setminus F \notin \mathcal{F}\}$, from which we have that if $\mathcal{F}$ is a family then $(\mathcal{F}^*)^* = \mathcal{F}$.

2.2.2. *Filter and Ramsey property.* If a family $\mathcal{F}$ is closed under finite intersections and is proper, then it is called a *filter*.

A family $\mathcal{F}$ has the *Ramsey property* if $A = A_1 \cup A_2 \in \mathcal{F}$ then $A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. It is well known that a proper family has the Ramsey property if and only if its dual $\mathcal{F}^*$ is a filter [13].

A subset S of $\mathbb{Z}$ is *syndetic* if it has a bounded gap, i.e. there is $N \in \mathbb{N}$ such that $\{i, i+1, \cdots, i+N\} \cap S \neq \emptyset$ for every $i \in \mathbb{Z}$. A subset S is an *IP-set*, if there is a subsequence $\{p_i\}$ of $\mathbb{Z}$ such that

$$S \supset \{p_{i_1} + \ldots + p_{i_n} : i_1 < \ldots < i_n, n \in \mathbb{N}\}.$$

It is known that the family of all IP^*-sets is a filter and each IP^*-set is syndetic [13].

The *upper Banach density* and *lower Banach density* of S are

$$BD^*(S) = \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|}, \text{ and } BD_*(S) = \liminf_{|I| \to \infty} \frac{|S \cap I|}{|I|},$$

where I ranges over intervals of $\mathbb{Z}$, while the *upper density* of S and the *lower density* of S are

$$D^*(S) = \limsup_{n \to \infty} \frac{|S \cap [-n, n]|}{2n+1}, \text{ and } D_*(S) = \liminf_{n \to \infty} \frac{|S \cap [-n, n]|}{2n+1}.$$

If $D^*(S) = D_*(S)$, then we say the *density* of S is $D(S) = D^*(S) = D_*(S)$.

2.3. **A Bergelson-Host-Kra' Theorem and a consequence.** In this subsection we explain how Bergelson-Host-Kra's result is related to Problem B-II. First we need some definitions.

Definition 2.1. Let $k \geq 1$ be an integer and let $X = G/\Gamma$ be a d-step nilmanifold. Let ϕ be a continuous real (or complex) valued function on X and let $a \in G$ and $b \in X$. The sequence $\{\phi(a^n \cdot b)\}$ is called a basic d-step nilsequence. A d-step nilsequence is a uniform limit of basic d-step nilsequences.

For the definition of nilmanifolds see Section 3.

Definition 2.2. Let $\{a_n : n \in \mathbb{Z}\}$ be a bounded sequence. We say that a_n tends to zero in uniform density, and we write UD-Lim $a_n = 0$, if

$$\lim_{N \longrightarrow +\infty} \sup_{M \in \mathbb{Z}} \sum_{n=M}^{M+N-1} |a_n| = 0.$$

Equivalently, UD-Lim $a_n = 0$ if and only if for any $\epsilon > 0$, the set $\{n \in \mathbb{Z} : |a_n| > \epsilon\}$ has upper Banach density zero. Now we state their result.

Theorem 2.3 (Bergelson-Host-Kra)**.** [4, Theorem 1.9] *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system, let $f \in L^\infty(\mu)$ and let $d \geq 1$ be an integer. The sequence $\{I_f(d, n)\}$ is the sum of a sequence tending to zero in uniform density and a d-step nilsequence, where*

$$I_f(d, n) = \int f(x) f(T^n x) \dots f(T^{dn} x)\ d\mu(x). \tag{2.1}$$

Especially, for any $A \in \mathcal{X}$

$$\{I_{1_A}(d, n)\} = \{\mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A)\} = F_d + N, \tag{2.2}$$

where F_d is a d-step nilsequence and N tending to zero in uniform density. Regard F_d as a function $F_d : \mathbb{Z} \to \mathbb{C}$. By [20] there is a d-step nilsystem (Z, S), $x_0 \in Z$ and a continuous function $\phi \in C(Z)$ such that

$$F_d(n) = \phi(S^n x_0).$$

We claim that $\phi(x_0) > 0$ if $\mu(A) > 0$. Assume that contrary that $\phi(x_0) \leq 0$. By [14] or [6, Theorem 6.15] there is $c > 0$ such that

$$\{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > c\}$$

is an IP^*-set. On the other hand there is a small neighborhood V of x_0 such that $\phi(x) < \frac{1}{2}c$ for each $x \in V$ by the continuity of ϕ. It is known that $N(x_0, V)$ is an IP^*-set [13] since (Z, S) is distal [2, Ch 4, Theorem 3] or [25]. This contradicts to (2.2) by the facts that the family of IP^*-sets is a filter, each IP^*-set is syndetic and $N(n)$ tends to zero in uniform density. That is, we have shown that $\phi(x_0) > 0$ if $\mu(A) > 0$.

Hence if $\mu(A) > 0$ then for each $\epsilon > 0$, $\{n \in \mathbb{Z} : \phi(S^n x_0) > \phi(x_0) - \frac{1}{2}\epsilon\}$ is a Nil$_d$-Bohr$_0$ set. Since $\{n \in \mathbb{Z} : |N(n)| > \frac{1}{2}\epsilon\}$ has zero upper Banach density we have the following corollary

Corollary 2.4. *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Then for all $A \in \mathcal{X}$ with $\mu(A) > 0$ and $\epsilon > 0$, the set*

$$I = \{n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \ldots \cap T^{-dn}A) > \phi(x_0) - \epsilon\}$$

is an almost Nil_d $Bohr_0$-set, i.e. there is some subset M with $BD^*(M) = 0$ such that $I \Delta M$ is a Nil_d $Bohr_0$-set.

It follows that problem B-II(2) has a positive answer ignoring a set with zero density, since for a minimal system (X, T), each invariant measure of (X, T) is fully supported.

2.4. **Furstenberg correspondence principle.** Let $\mathcal{F}(\mathbb{Z})$ denote the collection of finite non-empty subsets of $\mathbb{Z}$. It is well known that

Theorem 2.5 (Topological case). (1) *Let $E \subseteq \mathbb{Z}$ be a syndetic set. Then there exist a minimal system (X, T) and a non-empty open set $U \subseteq X$ such that*

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n\in\alpha} T^{-n}U \neq \emptyset\} \subseteq \{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n\in\alpha}(E - n) \neq \emptyset\}.$$

(2) *For any minimal system (X, T) and any open non-empty set U, there is a syndetic set E such that*

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n\in\alpha}(E - n) \neq \emptyset\} \subseteq \{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n\in\alpha} T^{-n}U \neq \emptyset\}.$$

Theorem 2.6 (Measurable case). (1) *Let $E \subseteq \mathbb{Z}$ with $BD^*(E) > 0$. Then there exists a measurable system $(X, \mathcal{X}, \mu, T)$ and $A \in \mathcal{X}$ with $\mu(A) = BD^*(E)$ such that for all $\alpha \in \mathcal{F}(\mathbb{Z})$*

$$BD^*(\bigcap_{n\in\alpha}(E - n)) \geq \mu(\bigcap_{n\in\alpha} T^{-n}A).$$

(2) *Let $(X, \mathcal{X}, \mu, T)$ be a measurable system and $A \in \mathcal{X}$ with $\mu(A) > 0$. There is a set E with $D^*(E) \geq \mu(A)$ such that*

$$\{\alpha \in \mathcal{F}(\mathbb{Z}) : \bigcap_{n\in\alpha}(E - n) \neq \emptyset\} \subseteq \{\alpha \in \mathcal{F}(\mathbb{Z}) : \mu(\bigcap_{n\in\alpha} T^{-n}U) > 0\}.$$

2.5. **Equivalence.** In this subsection we explain why Problems B-I,II,III are equivalent. Let $\mathcal{F}$ be the family generated by all sets of forms $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$, with (X, T) a minimal system, U a non-empty open subset of X. Then it is clear from the definition that

$$\mathcal{F}_{Bir_d} = \mathcal{F}^*.$$

Proposition 2.7. *For any $d \in \mathbb{N}$ the following statements are equivalent.*

(1) *For any Nil_d $Bohr_0$-set A, there is a syndetic subset S of $\mathbb{Z}$ with $A \supset \{n \in \mathbb{Z} : S \cap (S - n) \cap \ldots \cap (S - dn) \neq \emptyset\}$.*
(2) *For any Nil_d $Bohr_0$-set A, there are a minimal system (X, T) and a non-empty open subset U of X with $A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$.*
(3) $\mathcal{F}_{Bir_d} \subset \mathcal{F}^*_{d,0}$.

Proof. Let $d \in \mathbb{N}$ be fixed. (1)⇒(2). Let A be a Nil$_d$ Bohr$_0$-set, then there is a syndetic subset S of $\mathbb{Z}$ with $A \supset \{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$. For such S using Theorem 2.5, we get that there exist a minimal system (X,T) and a non-empty open set $U \subseteq X$ such that $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\} \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$. Thus $A \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$. (2)⇒(1) follows similarly by the above argument. (2)⇒(3) follows by the definition. (3)⇒(2). Since $\mathcal{F}_{Bir_d} \subset \mathcal{F}^*_{d,0}$ and $\mathcal{F}_{Bir_d} = \mathcal{F}^*$, we have that $\mathcal{F}^* \subset \mathcal{F}^*_{d,0}$ which implies that $\mathcal{F} \supset \mathcal{F}_{d,0}$. □

Proposition 2.8. *For any $d \in \mathbb{N}$ the following statements are equivalent.*

(1) *For any syndetic set S, $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$ is a Nil$_d$ Bohr$_0$-set.*
(2) *For any minimal system (X,T), and any open non-empty $U \subset X$, $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$ is a Nil$_d$ Bohr$_0$-set.*
(3) $\mathcal{F}_{Bir_d} \supset \mathcal{F}^*_{d,0}$.

Proof. Let $d \in \mathbb{N}$ be fixed. (1)⇒(2). Let (X,T) be a minimal system and U be a non-empty open set of X. By Theorem 2.5, there is a syndetic set S such that

$$\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\} \subset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}.$$

By (1), $\{n \in \mathbb{Z} : S \cap (S-n) \cap \ldots \cap (S-dn) \neq \emptyset\}$ is a Nil$_d$ Bohr$_0$-set, and so is $\{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}$. Similarly, we have (2)⇒(1). (2)⇒(3) follows by the definition. (3)⇒(2). By Corollary D we have $\mathcal{F}^*_{d,0} = \mathcal{F}^*$, i.e. $\mathcal{F}_{d,0} = \mathcal{F}$. □

2.6. **Other related problems.** We remark that similar problems can be formulated replacing syndetic sets by sets with positive Banach density, minimal systems by ergodic systems and open non-empty sets by positive measurable sets.

3. Nilsystems

In this section we recall some basic facts concerning nilpotent Lie groups and nilmanifolds. Since in the proofs of our main results we need to use the metric in the nilpotent matrix Lie group, we state some its basic properties. Note that we follow Green and Tao [15] to define such a metric.

3.1. Nilmanifolds and nilsystems.

3.1.1. *Nilpotent groups.* Let G be a group. For $g, h \in G$, we write $[g,h] = ghg^{-1}h^{-1}$ for the commutator of g and h and we write $[A,B]$ for the subgroup spanned by $\{[a,b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that G is *d-step nilpotent* if G_{d+1} is the trivial subgroup.

3.1.2. *Nilmanifolds.* Let G be a d-step nilpotent Lie group and Γ a discrete cocompact subgroup of G, i.e. a uniform subgroup of G. The compact manifold $X = G/\Gamma$ is called a *d-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g,x) \mapsto gx$. The Haar measure μ of X is the unique probability measure on X invariant under this action. Let $\tau \in G$ and T be the transformation

$x \mapsto \tau x$ of X, i.e the nilrotation induced by $\tau \in G$. Then (X, T, μ) is called a basic *d-step nilsystem*. See [10, 26] for the details.

3.1.3. *d-step nilsystem and system of order d.* We also make use of inverse limits of nilsystems and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $(X_i, T_i)_{i\in\mathbb{N}}$ are systems with $diam(X_i) \le M < \infty$ and $\phi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\varprojlim\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x, y) = \sum_{i\in\mathbb{N}} 1/2^i \rho_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

Definition 3.1. [Host-Kra-Maass] [19] A system (X, T) is called a *d-step nilsystem*, if it is an inverse limit of basic d-step nilsystems. A system (X, T) is called a *system of order d*, if it is a minimal d-step nilsystem, equivalently it is an inverse limit of basic d-step minimal nilsystems.

Recall that a subset $A \subseteq \mathbb{Z}$ is a *Nil_d $Bohr_0$-set* if there exist a d-step nilsystem (X, T), $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that $N(x_0, U)$ is contained in A. As each basic d-step nilsystem is distal, so is a d-step nilsystem. Hence by Definition 3.1, it is not hard to see that a subset $A \subseteq \mathbb{Z}$ is a *Nil_d $Bohr_0$-set* if and only if there exist a basic d-step (minimal) nilsystem (X, T) (or a *system* (X, T) *of order* d), $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that $N(x_0, U)$ is contained in A. Note that here we need the facts that the product of finitely many of d-step nilmanifolds is a d-step nilmanifold, and the orbit closure of any point in a basic d-step nilsystem is a d-step nilmanifold [25, Theorem 2.21].

3.2. **Reduction.** Let $X = G/\Gamma$ be a nilmanifold. Then there exists a connected, simply connected nilpotent Lie group $\widehat{G}$ and $\widehat{\Gamma} \subseteq \widehat{G}$ a co-compact subgroup such that X with the action of G is isomorphic to a submanifold $\widetilde{X}$ of $\widehat{X} = \widehat{G}/\widehat{\Gamma}$ representing the action of G in $\widehat{G}$. See [25] for more details.

Thus a subset $A \subseteq \mathbb{Z}$ is a *Nil_d $Bohr_0$-set* if and only if there exist a basic d-step nilsystem $(G/\Gamma, T)$ with G is a connected, simply connected nilpotent Lie group and Γ a co-compact subgroup of G, $x_0 \in X$ and an open set $U \subseteq X$ containing x_0 such that $N(x_0, U)$ is contained in A.

3.3. **Nilpotent Lie group and Mal'cev basis.**

3.3.1. We will make use of the Lie algebra $\mathfrak{g}$ of a d-step nilpotent Lie group G together with the exponential map $\exp : \mathfrak{g} \longrightarrow G$. When G is a connected, simply-connected d-step nilpotent Lie group the exponential map is a diffeomorphism [10, 26]. In particular, we have a logarithm map $\log : G \longrightarrow \mathfrak{g}$. Let

$$\exp(X * Y) = \exp(X)\exp(Y),\ X, Y \in \mathfrak{g}.$$

3.3.2. *Campbell-Baker-Hausdorff formula.* The following Campbell-Baker-Hausdorff formula (CBH formula) will be used frequently

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{p_i+q_i>0, 1\le i\le n} \frac{(\sum_{i=1}^n (p_i+q_i))^{-1}}{p_1!q_1!\dots p_n!q_n!} \\ \times (\mathrm{ad}\ X)^{p_1}(\mathrm{ad}\ Y)^{q_1}\dots(\mathrm{ad}\ X)^{p_n}(\mathrm{ad}\ Y)^{q_n-1}Y,$$

where $(\mathrm{ad}\ X)Y = [X, Y]$. (If $q_n = 0$, the term in the sum is $\dots(\mathrm{ad}\ X)^{p_n-1}X$; of course if $q_n > 1$, or if $q_n = 0$ and $p_n > 1$, then the term if zero.) The low order nonzero terms are well known,

$$\begin{aligned} X * Y =& X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] \\ & - \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] \\ & + (\text{ commutators in five or more terms}). \end{aligned}$$

3.3.3. We assume $\mathfrak{g}$ is the Lie algebra of G over $\mathbb{R}$, and $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map. The descending central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^{(1)} = \mathfrak{g};\ \mathfrak{g}^{(n+1)} = [\mathfrak{g}, \mathfrak{g}^{(n)}] = \mathrm{span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{g}^{(n)}\}.$$

Since $\mathfrak{g}$ is a d-step nilpotent Lie algebra, we have

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supsetneq \mathfrak{g}^{(2)} \supsetneq \dots \supsetneq \mathfrak{g}^{(d)} \supsetneq \mathfrak{g}^{(d+1)} = \{0\}.$$

We note that

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \forall i, j \in \mathbb{N}.$$

In particular, each $\mathfrak{g}^{(k)}$ is an ideal in $\mathfrak{g}$.

3.3.4. *Mal'cev Base.*

Definition 3.2. (Mal'cev base) Let G/Γ be an m-dimensional nilmanifold (i.e. G is a d-step nilpotent Lie group and Γ is a discrete uniform subgroup of G) and let $G = G_1 \supset \dots \supset G_d \supset G_{d+1} = \{e\}$ be the lower central series filtration. A basis $\mathcal{X} = \{X_1, \dots, X_m\}$ for the Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called a *Mal'cev basis* for G/Γ if the following four conditions are satisfied:

(1) For each $j = 0, \dots, m-1$ the subspace $\eta_j := \mathrm{Span}(X_{j+1}, \dots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_j := \exp \eta_j$ is a normal Lie subgroup of G.
(2) For every $0 < i < d$ we have $G_i = H_{l_{i-1}+1}$. Thus $0 = l_0 < l_1 < \dots < l_{d-1} \le m-2$.
(3) Each $g \in G$ can be written uniquely as $\exp(t_1X_1)\exp(t_2X_2)\dots\exp(t_mX_m)$, for $t_i \in \mathbb{R}$.
(4) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$.

Note that such a basis exists when G is a connected, simply connected d-step nilpotent Lie group [10, 15, 26].

3.4. **Metrics on nilmanifolds.** For a connected, simply connected d-step nilpotent Lie group G, we can use a Mal'cev basis X to put a metric structure on G and on G/Γ.

Definition 3.3 (Metrics on G and G/Γ)**.** [15] Let G/Γ be a nilmanifold with a Mal'cev basis $\mathcal{X}$, where G is a connected, simply connected Lie group and Γ is a discrete uniform subgroup of G. Let $\phi : G \longrightarrow \mathbb{R}^m$ with

$$g = \exp(t_1X_1)\dots\exp(t_mX_m) \mapsto (t_1,\dots,t_m).$$

We define $\rho = \rho_X : G \times G \longrightarrow \mathbb{R}$ to be the largest metric such that $\rho(x,y) \le |\phi(xy^{-1})|$ for all $x,y \in G$, where $|\cdot|$ denotes the ℓ^∞-norm on $\mathbb{R}^m$. More explicitly, we have

$$\rho(x,y) = \inf\Big\{\sum_{i=1}^n \min\{|\phi(x_{i-1}x_i^{-1})|, |\phi(x_ix_{i-1}^{-1})|\} : x_0,\dots,x_n \in G; x_0 = x, x_n = y\Big\}.$$

This descends to a metric on G/Γ by setting

$$\rho(x\Gamma, y\Gamma) := \inf\{d(x',y') : x',y' \in G; x' = x(\text{mod } \Gamma); y' = y(\text{mod } \Gamma)\}.$$

It turns out that this is indeed a metric on G/Γ, see [15]. Since ρ is right-invariant, we also have

$$\rho(x\Gamma, y\Gamma) = \inf_{\gamma\in\Gamma} \rho(x, y\gamma).$$

3.5. **Base points.** The following proposition should be well known.

Proposition 3.4. *Let $X = G/\Gamma$ be a nilmanifold, T be a nilrotation induced by $a \in G$. Let $x \in G$ and U be an open neighborhood $x\Gamma$ in X. Then there are a uniform subgroup $\Gamma_x \subset G$ and an open neighborhood $V \subset G/\Gamma_x$ of $e\Gamma_x$ such that*

$$N_T(x\Gamma, U) = N_{T'}(e\Gamma_x, V),$$

where T' is a nilrotation induced by $a \in G$ in $X' = G/\Gamma_x$.

Proof. Let $\Gamma_x = x\Gamma x^{-1}$. Then Γ_x is also a uniform subgroup of G.

Put $V = Ux^{-1}$, where we view U as the collections of equivalence classes. It is easy to see that $V \subset G/\Gamma_x$ is open, which contains $e\Gamma_x$. Let $n \in N_T(x\Gamma, U)$ then $a^n x\Gamma \in U$ which implies that $a^n x\Gamma x^{-1} \in Ux^{-1} = V$, i.e. $n \in N_{T'}(e\Gamma_x, V)$. The other direction follows similarly. $\square$

3.6. **Nilpotent Matrix Lie Group.**

3.6.1. Let $M_{d+1}(\mathbb{R})$ denote the space of all $(d+1)\times(d+1)$-matrices with real entries. For $A = (A_{ij})_{1\le i,j\le d+1} \in M_{d+1}(\mathbb{R})$, we define

$$\|A\| = \Big(\sum_{i,j=1}^{d+1} |A_{ij}|^2\Big)^{\frac{1}{2}}. \tag{3.1}$$

Then $\|\cdot\|$ is a norm on $M_{d+1}(\mathbb{R})$ and the norm satisfies the inequalities

$$\|A+B\| \le \|A\| + \|B\| \text{ and } \|AB\| \le \|A\|\|B\|$$

for $A, B \in M_{d+1}(\mathbb{R})$.

3.6.2. Let $\mathbf{a} = (a_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$. Then corresponding to $\mathbf{a}$ we define $\mathbf{M}(\mathbf{a})$ with

$$\mathbf{M}(\mathbf{a}) = \begin{pmatrix} 1 & a_1^1 & a_1^2 & a_1^3 & \dots & a_1^{d-1} & a_1^d \\ 0 & 1 & a_2^1 & a_2^2 & \dots & a_2^{d-2} & a_2^{d-1} \\ 0 & 0 & 1 & a_3^1 & \dots & a_3^{d-3} & a_3^{d-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{d-1}^1 & a_{d-1}^2 \\ 0 & 0 & 0 & 0 & \dots & 1 & a_d^1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

3.6.3. Let $\mathbb{G}_d$ be the (full) upper triangular group

$$\mathbb{G}_d = \{\mathbf{M}(\mathbf{a}) : a_i^k \in \mathbb{R}, 1 \le k \le d, 1 \le i \le d-k+1\}.$$

The group $\mathbb{G}_d$ is a d-step nilpotent group, and it is clear that for $A \in \mathbb{G}_d$ there exists a unique $\mathbf{c} = (c_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ such that $A = \mathbf{M}(\mathbf{c})$. Let

$$\Gamma = \{\mathbf{M}(\mathbf{h}) : h_i^k \in \mathbb{Z}, 1 \le k \le d, 1 \le i \le d-k+1\}.$$

Then Γ is a uniform subgroup of $\mathbb{G}_d$.

3.6.4. Let $\mathbf{a} = (a_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ and $\mathbf{b} = (b_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$. If $\mathbf{c} = (c_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ such that $\mathbf{M}(\mathbf{c}) = \mathbf{M}(\mathbf{a})\mathbf{M}(\mathbf{b})$, then

$$c_i^k = \sum_{j=0}^{k} a_i^{k-j} b_{i+k-j}^j = a_i^k + \Big(\sum_{j=1}^{k-1} a_i^{k-j} b_{i+k-j}^j\Big) + b_i^k \tag{3.2}$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$, where we assume $a_1^0 = a_2^0 = \ldots = a_d^0 = 1$ and $b_1^0 = b_2^0 = \ldots = b_d^0 = 1$.

3.6.5. Now we endow a compatible metric ρ on $\mathbb{G}_d$ and $\mathbb{G}_d/\Gamma$.

Definition 3.5 (Metric on $\mathbb{G}_d$)**.** Let I be the $(d+1)\times(d+1)$ identity matrix. Define $\rho : \mathbb{G}_d \times \mathbb{G}_d \longrightarrow \mathbb{R}$ such that

$$\rho(A,B) = \inf\{\sum_{i=1}^{n} \min\{\|A_{i-1}A_i^{-1} - I\|, \|A_iA_{i-1}^{-1} - I\|\} : A_0, \ldots, A_n \in G; A_0 = A, A_n = B\}.$$

Lemma 3.6. *For any $A, B \in \mathbb{G}_d$,*

$$\rho(A,B) \le \|AB^{-1} - I\| \le \|A-B\|\|B^{-1}\| \text{ and}$$
$$\frac{\|A-B\|}{\|B\|} \le \|AB^{-1} - I\| \le e^{\rho(A,B)+\rho(A,B)^2+\cdots+\rho(A,B)^d} - 1.$$

Proof. It is clear that it is sufficient to prove

$$\|AB^{-1} - I\| \le e^{\rho(A,B)+\rho(A,B)^2+\cdots+\rho(A,B)^d} - 1 \tag{3.3}$$

for $A, B \in \mathbb{G}_d$. The others are obvious. Let $A, B \in \mathbb{G}_d$. For any $\epsilon > 0$, there exist $A_0, \dots, A_n \in \mathbb{G}_d; A_0 = A, A_n = B$ such that

$$\sum_{i=1}^{n} \min\{\|A_{i-1}A_i^{-1} - I\|, \|A_i A_{i-1}^{-1} - I\|\} \le \rho(A, B) + \epsilon.$$

Let $t_i = \min\{\|A_{i-1}A_i^{-1} - I\|, \|A_i A_{i-1}^{-1} - I\|\}$ for $i = 1, 2, \cdots, n$. Note that for $C \in \mathbb{G}_d$, $C^{-1} = I + \sum_{i=1}^{d}(I - C)^i$. Hence

$$\|C^{-1} - I\| \le \sum_{i=1}^{d} \|C - I\|^i.$$

Moreover, if we set $t = \min\{\|C - I\|, \|C^{-1} - I\|\}$, then

$$\|C^{-1} - I\| \le t(1 + t + t^2 + \cdots + t^{d-1}). \tag{3.4}$$

Then by (3.4),

$$\begin{aligned} \|A_{i-1}A_i^{-1} - I\| &\le t_i(1 + t_i + t_i^2 + \cdots + t_i^{d-1}) \\ &\le t_i(1 + (\rho(A, B) + \epsilon) + \cdots + (\rho(A, B) + \epsilon)^{d-1}) \end{aligned}$$

for $i = 1, 2, \cdots, n$. Thus

$$\begin{aligned} \sum_{i=1}^{n} \|A_{i-1}A_i^{-1} - I\| &\le \sum_{i=1}^{n} t_i(1 + (\rho(A, B) + \epsilon) + \cdots + (\rho(A, B) + \epsilon)^{d-1}) \\ &\le (\rho(A, B) + \epsilon) + \cdots + (\rho(A, B) + \epsilon)^d. \end{aligned}$$

Now

$$\begin{aligned} 1 + \|AB^{-1} - I\| &= 1 + \|A_0 A_n^{-1} - I\| = 1 + \|A_0 A_1^{-1} A_1 A_n^{-1} - I\| \\ &= 1 + \|(A_0 A_1^{-1} - I)(A_1 A_n^{-1} - I) + (A_0 A_1^{-1} - I) + (A_1 A_n^{-1} - I)\| \\ &\le 1 + \|(A_0 A_1^{-1} - I)\| \|(A_1 A_n^{-1} - I)\| + \|(A_0 A_1^{-1} - I)\| + \|(A_1 A_n^{-1} - I)\| \\ &= (1 + \|A_0 A_1^{-1} - I\|)(1 + \|A_1 A_n^{-1} - I\|). \end{aligned}$$

Continuing this process we get that

$$\begin{aligned} 1 + \|AB^{-1} - I\| &\le (1 + \|A_0 A_1^{-1} - I\|)(1 + \|A_1 A_2^{-1} - I\|) \cdots (1 + \|A_{n-1} A_n^{-1} - I\|) \\ &\le e^{\|A_0 A_1^{-1} - I\| + \|A_1 A_2^{-1} - I\| + \cdots + \|A_{n-1} A_n^{-1} - I\|} \\ &= e^{(\rho(A,B)+\epsilon) + \cdots + (\rho(A,B)+\epsilon)^d} \end{aligned}$$

This implies $\|AB^{-1} - I\| \le e^{(\rho(A,B)+\epsilon)+\cdots+(\rho(A,B)+\epsilon)^d} - 1$. Let $\epsilon \searrow 0$, we get (3.3). This ends the proof of the lemma. □

Proposition 3.7 (Metrics on $\mathbb{G}_d$ and $\mathbb{G}_d/\Gamma$)**.** *ρ is a right-invariant metric on $\mathbb{G}_d$. This descends to a metric on $\mathbb{G}_d/\Gamma$ by setting*

$$\rho(A\Gamma, B\Gamma) := \inf\{\rho(A\gamma, B\gamma') : \gamma, \gamma' \in \Gamma\}.$$

Since ρ is right-invariant, we also have

$$\rho(A\Gamma, B\Gamma) = \inf_{\gamma \in \Gamma} \rho(A, B\gamma).$$

Proof. Firstly, it is clear that $\rho : \mathbb{G}_d \times \mathbb{G}_d \longrightarrow \mathbb{R}$ is a right-invariant, non-negative function and for $A, B, C \in \mathbb{G}_d$,

$$\rho(A,B) = \rho(B,A) \text{ and } \rho(A,C) \le \rho(A,B) + \rho(B,C).$$

By Lemma 3.6 $\rho(A,B) = 0$ if and only if $\|A - B\| = 0$, i.e., $A = B$. Thus ρ is a right-invariant metric on $\mathbb{G}_d$. Moreover by Lemma 3.6, we know that the metric ρ is equivalent to the metric induced by the norm $\|\cdot\|$ on $\mathbb{G}_d$. Thus, ρ is a compatible metric with topology of $\mathbb{G}_d$.

Next we are going to show that this descends to a metric on $\mathbb{G}_d/\Gamma$ by setting

$$\rho(A\Gamma, B\Gamma) := \inf\{\rho(A\gamma, B\gamma') : \gamma, \gamma' \in \Gamma\}.$$

Since ρ is a right-invariant metric on $\mathbb{G}_d$, it is sufficient to show that if $\rho(A\Gamma, B\Gamma) = 0$ then $A\Gamma = B\Gamma$. Suppose $\rho(A\Gamma, B\Gamma) = 0$. Since ρ is right-invariant, $\inf_{\gamma \in \Gamma} \rho(A\gamma, B) = 0$. Moreover we can find $\gamma_i \in \Gamma$ such that $\|B\|(e^{\rho(A\gamma_i,B)+\cdots+\rho(A\gamma_i,B)^d} - 1) < \frac{1}{2^i(1+\|A^{-1}\|)}$ for each $i \in \mathbb{N}$. By Lemma 3.6, we have

$$\|A\gamma_i - B\| \le \|B\|(e^{\rho(A\gamma_i,B)+\cdots+\rho(A\gamma_i,B)^d} - 1) < \frac{1}{2^i(1 + \|A^{-1}\|)}$$

for $i \in \mathbb{N}$. Thus for all $i \le j \in \mathbb{N}$,

$$\begin{aligned}
\|\gamma_i - \gamma_j\| &= \|A^{-1}(A(\gamma_i - B) - A(\gamma_j - B))\| \\
&\le \|A^{-1}\|(\|A\gamma_i - B\| + \|A\gamma_j - B\|) \\
&< \|A^{-1}\|\left(\frac{1}{2^i(1 + \|A^{-1}\|)} + \frac{1}{2^i(1 + \|A^{-1}\|)}\right) < 1.
\end{aligned}$$

Since $\gamma_i, \gamma_j \in \Gamma$, this implies $\gamma_i = \gamma_j$ for $i, j \in \mathbb{N}$. Thus

$$\|A\gamma_1 - B\| = \|A\gamma_j - B\| < \frac{1}{2^j(1 + \|A^{-1}\|)}$$

for any $j \in \mathbb{N}$. Hence $\|A\gamma_1 - B\| = 0$. So $A\gamma_1 = B$ and $A\Gamma = B\Gamma$. This ends the proof of the proposition. $\square$

4. Generalized polynomials

In this section we introduce the notions and basic properties of (special) generalized polynomials. It will be used in the following sections.

4.1. **Definitions.**

4.1.1. For a real number $a \in \mathbb{R}$, let $||a|| = \inf\{|a - n| : n \in \mathbb{Z}\}$ and

$$\lceil a \rceil = \min\{m \in \mathbb{Z} : |a - m| = ||a||\}.$$

When studying $\mathcal{F}_{d,0}$ we find that the generalized polynomials appear naturally. Here is the precise definition. Note that we use $f(n)$ or f to denote the generalized polynomials.

4.1.2. *Generalized polynomials.*

Definition 4.1. Let $d \in \mathbb{N}$. We define the *generalized polynomials* of degree $\le d$ (denoted by GP_d) by induction. For $d = 1$, GP_1 is the collection of functions from $\mathbb{Z}$ to $\mathbb{R}$ containing h_a, $a \in \mathbb{R}$ with $h_a(n) = an$ for each $n \in \mathbb{Z}$ which is closed under taking $\lceil\ \rceil$, multiplying by a constant and the finite sums.

Assume that GP_i is defined for $i < d$. Then GP_d is the collection of functions from $\mathbb{Z}$ to $\mathbb{R}$ containing GP_i with $i < d$, functions of the forms

$$a_0 n^{p_0} \lceil f_1(n) \rceil \dots \lceil f_k(n) \rceil$$

(with $a_0 \in \mathbb{R}$, $p_0 \ge 0$, $k \ge 0$, $f_l \in \mathrm{GP}_{p_l}$ and $\sum_{l=0}^{k} p_l = d$), which is closed under taking $\lceil\ \rceil$, multiplying by a constant and the finite sums. Let $\mathrm{GP}= \cup_{i=1}^{\infty}\mathrm{GP}_i$.

For example, $a_1\lceil a_2\lceil a_3 n\rceil\rceil + b_1 n \in \mathrm{GP}_1$, and $a_1\lceil a_2 n^2\rceil + b_1\lceil b_2\lceil b_3 n\rceil\rceil + c_1 n^2 + c_2 n \in \mathrm{GP}_2$, where $a_i, b_i, c_i \in \mathbb{R}$. Note that if $f \in \mathrm{GP}$ then $f(0) = 0$.

4.1.3. *Special generalized polynomials.* Since generalized polynomials are very complicated, we will specify a subclass of them, called the *special generalized polynomials* which will be used in our proofs of the main results. To do this, we need some notions.

For $a \in \mathbb{R}$, we define $L(a) = a$. For $a_1, a_2 \in \mathbb{R}$ we define $L(a_1, a_2) = a_1\lceil L(a_2)\rceil$. Inductively, for $a_1, a_2, \cdots, a_\ell \in \mathbb{R}$ ($\ell \ge 2$) we define

$$L(a_1, a_2, \cdots, a_\ell) = a_1\lceil L(a_2, a_3, \cdots, a_\ell)\rceil. \tag{4.1}$$

For example, $L(a_1, a_2, a_3) = a_1\lceil a_2\lceil a_3\rceil\rceil$.

We give now the precise definition of special generalized polynomials.

Definition 4.2. For $d \in \mathbb{N}$ we define *special generalized polynomials of degree* $\le d$, denoted by SGP_d as follows. SGP_d is the collection of generalized polynomials of the forms $L(n^{j_1}a_1, \cdots, n^{j_\ell}a_\ell)$, where $1 \le \ell \le d, a_1, \cdots, a_\ell \in \mathbb{R}, j_1, \cdots, j_\ell \in \mathbb{N}$ with $\sum_{t=1}^{\ell} j_t \le d$.

Thus $\mathrm{SGP}_1 = \{an : a \in \mathbb{R}\}$, $\mathrm{SGP}_2 = \{an^2, bn\lceil cn\rceil, en : a, b, c, e \in \mathbb{R}\}$ and SGP_3=$\mathrm{SGP}_2 \cup \{an^3, an\lceil bn^2\rceil, an^2\lceil bn\rceil, an\lceil bn\lceil cn\rceil\rceil : a, b, c \in \mathbb{R}\}$.

4.1.4. $\mathcal{F}_{GP_d}$ *and* $\mathcal{F}_{SGP_d}$. Let $\mathcal{F}_{GP_d}$ be the family generated by the sets of forms

$$\bigcap_{i=1}^{k} \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where $k \in \mathbb{N}$, $P_i \in GP_d$, and $\epsilon_i > 0$, $1 \le i \le k$. Note that $P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)$ if and only if $||P_i(n)|| < \epsilon_i$.

Let $\mathcal{F}_{SGP_d}$ be the family generated by the sets of forms

$$\bigcap_{i=1}^{k} \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\},$$

where $k \in \mathbb{N}$, $P_i \in SGP_d$, and $\epsilon_i > 0$, $1 \le i \le k$. Note that from the definition both $\mathcal{F}_{GP_d}$ and $\mathcal{F}_{SGP_d}$ are filters; and $\mathcal{F}_{SGP_d} \subset \mathcal{F}_{GP_d}$.

4.2. **Basic properties of generalized polynomials.**

4.2.1. The following lemmas lead the way to simplify the generalized polynomials. Note that for $f \in \mathrm{GP}$ we let $f^* = -\lceil f \rceil$.

Lemma 4.3. *Let $c \in \mathbb{R}$ and $f_1, \ldots, f_k \in GP$ with $k \in \mathbb{N}$. Then*

$$c\lceil f_1 \rceil \ldots \lceil f_k \rceil = c(-1)^k \prod_{i=1}^{k} (f_i - \lceil f_i \rceil) - c(-1)^k \sum_{\substack{i_1,\ldots,i_k \in \{1,*\} \\ (i_1,\ldots,i_k) \neq (*,\ldots,*)}} f_1^{i_1} \ldots f_k^{i_k}.$$

Particularly if $k = 2$ we get that

$$c\lceil f_1 \rceil \lceil f_2 \rceil = cf_1 \lceil f_2 \rceil - cf_1 f_2 + cf_2 \lceil f_1 \rceil + c(f_1 - \lceil f_1 \rceil)(f_2 - \lceil f_2 \rceil).$$

Proof. Expanding $\prod_{i=1}^{k} (f_i - \lceil f_i \rceil)$ we get that

$$\prod_{i=1}^{k} (f_i - \lceil f_i \rceil) = \sum_{i_1,\ldots,i_k \in \{1,*\}} f_1^{i_1} \ldots f_k^{i_k}.$$

So we have

$$c\lceil f_1 \rceil \ldots \lceil f_k \rceil = c(-1)^k \prod_{i=1}^{k} (f_i - \lceil f_i \rceil) - c(-1)^k \sum_{\substack{i_1,\ldots,i_k \in \{1,*\} \\ (i_1,\ldots,i_k) \neq (*,\ldots,*)}} f_1^{i_1} \ldots f_k^{i_k}.$$

$\square$

Let $c = 1$ in Lemma 4.3 we have

Lemma 4.4. *Let $f_1, f_2, \ldots, f_k \in GP$. Then*

$$f_1 \lceil f_2 \rceil \ldots \lceil f_k \rceil = (-1)^{k-1} \prod_{i=1}^{k} (f_i - \lceil f_i \rceil) + (-1)^k \sum_{\substack{i_1,\ldots,i_k \in \{1,*\} \\ (i_1,\ldots,i_k) \neq (1,*,\ldots,*)}} f_1^{i_1} \ldots f_k^{i_k}.$$

Particularly if $k = 2$ we have

$$f_1 \lceil f_2 \rceil = \lceil f_1 \rceil \lceil f_2 \rceil + f_1 f_2 - f_2 \lceil f_1 \rceil - (f_1 - \lceil f_1 \rceil)(f_2 - \lceil f_2 \rceil).$$

Let $k = 1$ in Lemma 4.3 we have

Lemma 4.5. *Let $c \in \mathbb{R}$ and $f \in GP$. Then $c\lceil f \rceil = cf - c(f - \lceil f \rceil)$.*

4.2.2. In the next subsection we will show that $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$. To do this we use induction. To make the proof clearer, first we give some results under the assumption

$$\mathcal{F}_{GP_{d-1}} \subset \mathcal{F}_{SGP_{d-1}}. \tag{4.2}$$

Definition 4.6. Let $r \in \mathbb{N}$ with $r \geq 2$. We define

$$\mathcal{SW}_r = \{\prod_{i=1}^{\ell} (w_i(n) - \lceil w_i(n) \rceil) : \ell \geq 2, r_i \geq 1, w_i(n) \in GP_{r_i} \text{ and } \sum_{i=1}^{\ell} r_i \leq r\}$$

and

$$\mathcal{W}_r = \mathbb{R} - \mathrm{Span}\{\mathcal{SW}_r\},$$

that is,

$$\mathcal{W}_r = \{\sum_{j=1}^{\ell} a_j p_j(n) : \ell \geq 1, a_j \in \mathbb{R}, p_j(n) \in \mathcal{SW}_r \text{ for each } j = 1, 2, \cdots, \ell\}.$$

Lemma 4.7. *Under the assumption* (4.2)*, one has for any* $p(n) \in \mathcal{W}_d$ *and* $\epsilon > 0$*,*

$$\{n \in \mathbb{Z} : p(n) \ (mod\ \mathbb{Z}) \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{SGP_{d-1}}.$$

Proof. Since $\mathcal{F}_{SGP_d}$ is a filter, it is sufficient to show that for any $p(n) = aq(n)$ and $\frac{1}{2} > \delta > 0$ with $q(n) \in \mathcal{SW}_d$ and $a \in \mathbb{R}$,

$$\{n \in \mathbb{Z} : p(n) \ (\text{mod } \mathbb{Z}) \in (-\delta, \delta)\} \in \mathcal{F}_{SGP_{d-1}}.$$

Note that as $q(n) \in \mathcal{SW}_d$, there exist $\ell \geq 2, r_i \geq 1, w_i(n) \in GP_{r_i}$ and $\sum_{i=1}^{\ell} r_i \leq d$ such that $q(n) = \prod_{i=1}^{\ell}(w_i(n) - \lceil w_i(n) \rceil)$. Since $\ell \geq 2$, one has $r_1 \leq d-1$ and so $w_1(n) \in GP_{d-1}$. By the assumption (4.2), $\{n \in \mathbb{Z} : w_1(n) \ (\text{mod } \mathbb{Z}) \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \in \mathcal{F}_{SGP_{d-1}}$. By the inequality $|q(n)| \leq |a||w_1(n) - \lceil w_1(n) \rceil|$ for $n \in \mathbb{Z}$, we get that

$$\begin{aligned}\{n \in \mathbb{Z} : p(n) \ (\text{mod } \mathbb{Z}) \in (-\delta, \delta)\} &\supset \{n \in \mathbb{Z} : |w_1(n) - \lceil w_1(n) \rceil| \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \\ &= \{n \in \mathbb{Z} : w_1(n) \ (\text{mod } \mathbb{Z}) \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\}.\end{aligned}$$

Thus $\{n \in \mathbb{Z} : p(n) \ (\text{mod } \mathbb{Z}) \in (-\delta, \delta)\} \in \mathcal{F}_{SGP_{d-1}}$ since $\{n \in \mathbb{Z} : w_1(n) \ (\text{mod } \mathbb{Z}) \in (-\frac{\delta}{1+|a|}, \frac{\delta}{1+|a|})\} \in \mathcal{F}_{SGP_{d-1}}$. □

Definition 4.8. Let $r \in \mathbb{N}$ with $r \geq 2$. For $q_1(n), q_2(n) \in GP_r$ we define

$$q_1(n) \simeq_r q_2(n)$$

if there exist $h_1(n) \in GP_{r-1}$ and $h_2(n) \in \mathcal{W}_r$ such that

$$q_2(n) = q_1(n) + h_1(n) + h_2(n) \ (\text{mod } \mathbb{Z})$$

for all $n \in \mathbb{Z}$.

Lemma 4.9. *Let* $p(n) \in GP_r$ *and* $q(n) \in GP_t$*,* $r, t \in \mathbb{N}$*. Then*

(1) $p(n)\lceil q(n) \rceil \simeq_{r+t} (p(n) - \lceil p(n) \rceil)q(n)$.

(2) *if* $q_1(n), q_2(n), \cdots, q_k(n) \in GP_t$ *such that* $q(n) = \sum_{i=1}^k q_i(n)$*, then*

$$p(n)\lceil q(n) \rceil \simeq_{r+t} \sum_{i=1}^{k} p(n)\lceil q_i(n) \rceil.$$

Proof. (1) follows from Lemma 4.4 and (2) follows from (1). □

Definition 4.10. For $r \in \mathbb{N}$, we define

$$GP'_r = \{p \in GP_r : \{n \in \mathbb{Z} : p(n) \ (\text{mod } \mathbb{Z}) \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{SGP_r} \text{ for any } \epsilon > 0\}.$$

Proposition 4.11. *It is clear that*

(1) *For* $p(n) \in GP_r$, $p(n) \in GP'_r$ *if and only if* $-p(n) \in GP'_r$.
(2) *If* $p_1(n), p_2(n), \cdots, p_k(n) \in GP'_r$ *then*

$$p(n) = p_1(n) + p_2(n) + \cdots + p_k(n) \in GP'_r.$$

(3) $\mathcal{F}_{GP_d} \subset \mathcal{F}_{SGP_d}$ *if and only if* $GP'_d = GP_d$.

Proof. (1) can be verified directly. (2) follows from the fact that for each $\epsilon > 0$, $\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \supset \cap_{i=1}^k \{n \in \mathbb{Z} : p_i(n) \pmod{\mathbb{Z}} \in (-\epsilon/k, \epsilon/k)\}$. (3) follows from the definition of GP'_d. $\square$

Lemma 4.12. *Let* $p(n), q(n) \in GP_d$ *with* $p(n) \simeq_d q(n)$. *Under the assumption* (4.2), $p(n) \in GP'_d$ *if and only if* $q(n) \in GP'_d$.

Proof. It follows from Lemma 4.7 and the fact that $\mathcal{F}_{SGP_d}$ is a filter. $\square$

4.3. $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$.

Theorem 4.13. $\mathcal{F}_{GP_d} = \mathcal{F}_{SGP_d}$.

Proof. It is easy to see that $\mathcal{F}_{SGP_d} \subset \mathcal{F}_{GP_d}$. So it remains to show $\mathcal{F}_{GP_d} \subset \mathcal{F}_{SGP_d}$. That is, if $A \in \mathcal{F}_{GP_d}$ then there is $A' \in \mathcal{F}_{SGP_d}$ with $A \supset A'$. We will use induction to show the proposition.

Assume first $d = 1$. In this case we let $GP_1(0) = \{g_a : a \in \mathbb{R}\}$, where $g_a(n) = an$ for each $n \in \mathbb{Z}$. Inductively if $GP_1(0), \ldots, GP_1(k)$ have been defined then $f \in GP_1(k+1)$ if and only if $f \in GP_1 \setminus (\bigcup_{j=0}^k GP_1(j))$ and there are $k+1$ $\lceil \ \rceil$ in f. It is clear that $GP_1 = \cup_{k=1}^\infty GP_1(k)$. If $f \in GP_1(0)$ then it is clear that $f \in GP'_1$. Assume that $GP_1(0), \ldots, GP_1(k) \subset GP'_1$ for some $k \in \mathbb{Z}_+$.

Let $f \in GP_1(k+1)$. We are going to show that $f \in GP'_1$. If $f = f_1 + f_2$ with $f_1, f_2 \in \bigcup_{i=0}^k GP_1(i)$, then by the above assumption and Proposition 4.11 we conclude that $f \in GP'_1$. The remaining case is $f = c\lceil f_1 \rceil + f_2$ with $c \in \mathbb{R} \setminus \{0\}$, $f_1 \in GP_1(k)$, and $f_2 \in GP_1(0)$. By Proposition 4.11 and the fact $GP_1(0) \subseteq GP'_1$, $f \in GP'_1$ if and only if $c\lceil f_1 \rceil \in GP'_1$. So it remains to show $c\lceil f_1 \rceil \in GP'_1$. By Lemma 4.5 we have $c\lceil f_1 \rceil = cf_1 - c(f_1 - \lceil f_1 \rceil)$. It is clear that $cf_1 \in GP_1(k) \subset GP'_1$ since $f_1 \in GP_1(k) \subset GP'_1$. For any $\epsilon > 0$ since

$$\{n \in \mathbb{Z} : || - c(f_1(n) - \lceil f_1(n) \rceil)|| < \epsilon\} \supset \Big\{n \in \mathbb{Z} : ||f_1(n)|| < \frac{\epsilon}{1 + |c|}\Big\},$$

it implies that $-c(f_1 - \lceil f_1 \rceil) \in GP'_1$. By Proposition 4.11 again we conclude that $c\lceil f_1 \rceil \in GP'_1$. Hence $f \in GP'_1$. Thus $GP_1 \subseteq GP'_1$ and we are done for the case $d = 1$ by Proposition 4.11 (3).

Assume that we have proved $\mathcal{F}_{GP_{d-1}} \subset \mathcal{F}_{SGP_{d-1}}$ $d \geq 2$, i.e. the assumption (4.2) holds. We define $GP_d(k)$ with $k = 0, 1, 2, \ldots$. First $f \in GP_d(0)$ if and only if there is no $\lceil \ \rceil$ in f, i.e. f is the usual polynomial of degree $\leq d$. Inductively if $GP_d(0), \ldots, GP_d(k)$ have been defined then $f \in GP_{k+1}$ if and only if $f \in GP_d \setminus (\bigcup_{j=0}^k GP_d(j))$ and there are $k+1$ $\lceil \ \rceil$ in f. It is clear that $GP_d = \cup_{k=0}^\infty GP_d(k)$. We now show $GP_d(k) \subseteq GP'_d$ by induction on k.

Let f be a usual polynomial of degree $\le d$. Then $f(n) = a_0n^d + f_1(n) \simeq_d a_0n^d$ with $f_1 \in GP_{d-1}$. By Lemma 4.12, $f \in GP'_d$ since $a_0n^d \in \mathrm{SGP}_d \subset GP'_d$. This shows $GP_d(0) \subset GP'_d$. Now assume that for some $k \in \mathbb{Z}_+$ we have proved

$$\bigcup_{i=0}^{k} GP_d(i) \subseteq GP'_d. \tag{4.3}$$

Let $f \in GP_d(k+1)$. We are going to show that $f \in GP'_d$. If $f = f_1 + f_2$ with $f_1, f_2 \in \bigcup_{i=0}^k GP_d(i)$, then by the assumption (4.3) and Proposition 4.11 (2) we conclude that $f \in GP'_d$. The remaining case is that f can be expressed as the sum of a function in $GP_d(0)$ and a function $g \in GP_d(k+1)$ having the form of

(1) $g = c\lceil f_1\rceil \dots \lceil f_l\rceil$ with $c \neq 0$, $l \ge 1$ or
(2) $g = g_1(n)\lceil g_2(n)\rceil \dots \lceil g_l(n)\rceil$ for any $n \in \mathbb{Z}$ with $g_1(n) \in SGP_r$ and $r < d$.

Since $GP_d(0) \subset GP'_d$, $f \in GP'_d$ if and only if $g \in GP'_d$ by Proposition 4.11. It remains to show that $g \in GP'_d$. There are two cases.

Case (1): $g = c\lceil f_1\rceil \dots \lceil f_l\rceil$ with $c \neq 0$, $l \ge 1$.

If $l = 1$, then $g = c\lceil f_1\rceil$ with $f_1 \in GP_d(k)$. By Lemma 4.5 we have $c\lceil f_1\rceil = cf_1 - c(f_1 - \lceil f_1\rceil)$. It is clear that $cf_1 \in GP_d(k) \subset GP'_d$ since $f_1 \in GP_d(k) \subset GP'_d$. For any $\epsilon > 0$ since

$$\{n \in \mathbb{Z} : ||-c(f_1(n) - \lceil f_1(n)\rceil)|| < \epsilon\} \supset \Big\{n \in \mathbb{Z} : ||f_1(n)|| < \frac{\epsilon}{1+|c|}\Big\},$$

it implies that $-c(f_1 - \lceil f_1\rceil) \in GP'_d$. By Proposition 4.11 again we conclude that $g = c\lceil f_1\rceil \in GP'_d$.

If $l \ge 2$, using Lemmas 4.3 and 4.7 we get that

$$c\lceil f_1\rceil \dots \lceil f_l\rceil \simeq_d -c(-1)^l \sum_{\substack{i_1,\dots,i_l \in \{1,*\} \\ (i_1,\dots,i_l) \neq (*,\dots,*)}} f_1^{i_1} \dots f_l^{i_l}.$$

Since each term of the right side is in $GP_d(k)$, $g \in GP'_d$ by Lemma 4.12, the assumption (4.3) and Proposition 4.11 (2).

Case (2): $g = g_1(n)\lceil g_2(n)\rceil \dots \lceil g_l(n)\rceil$ for any $n \in \mathbb{Z}$ with $g_1 \in SGP_r$ and $1 \le r < d$.

In this case using Lemmas 4.4 and 4.7 we get that

$$g_1\lceil g_2\rceil \dots \lceil g_l\rceil \simeq_d (-1)^l \sum_{\substack{i_1,\dots,i_l \in \{1,*\} \\ (i_1,\dots,i_l) \neq (1,*,\dots,*),(*,*,\dots,*)}} g_1^{i_1} \dots g_l^{i_l}.$$

Assume $i_1, \dots, i_l \in \{1,*\}$ with $(i_1, \dots, i_l) \neq (1,*,\dots,*), (*,*,\dots,*)$. If there are at least two 1 appearing in $(i_1, i_2, \cdots, i_l)$, then $(-1)^\ell g_1^{i_1} \dots g_l^{i_\ell} \in \bigcup_{i=0}^k GP_d(i)$. Hence

$$(-1)^\ell g_1^{i_1} \dots g_l^{i_\ell} \in GP'_d$$

by the assumption (4.3). The remaining situation is that $i_1 = *$ and there is exact one 1 appearing in $(i_2, \dots, i_l)$. In this case, $(-1)^\ell g_1^{i_1} \dots g_l^{i_\ell} \in GP_d(k+1)$ is the finite sum of the forms $a_1n^{t_1}\lceil h_1(n)\rceil \dots \lceil h_{l'_1}(n)\rceil$ with $t_1 \ge 1$ and $h_1(n) = g_1(n)$; or the forms $c\lceil h_l\rceil \dots \lceil h_{l_1}\rceil$ or terms in GP'_d.

If the term has the form $a_1 n^{t_1}\lceil h_1(n)\rceil \dots \lceil h_{l'_1}(n)\rceil$ with $t_1 \ge 1$ and $h_1(n) = g_1(n)$, we let $g_1^{(1)}(n) = a_1 n^{t_1}\lceil h_1(n)\rceil = a_1 n^{t_1}\lceil g_1(n)\rceil \in SGP_{r_1}$. It is clear $d \ge r_1 > r$. If $r_1 = d$, then $a_1 n^{t_1}\lceil h_1(n)\rceil \dots \lceil h_{l'_1}(n)\rceil = g_1^1(n) \in GP'_d$ since $SGP_d \subset GP'_d$. If $r_1 < d$, then we write

$$a_1 n^{t_1}\lceil h_1(n)\rceil \dots \lceil h_{l'_1}(n)\rceil = g_1^{(1)}(n)\lceil g_2^{(1)}(n)\rceil \dots \lceil g_{l_1}^{(1)}(n)\rceil.$$

By using Case (1) we conclude that

$g \simeq_d$ finite sum of the forms $g_1^{(1)}(n)\lceil g_2^{(1)}(n)\rceil \dots \lceil g_{l_1}^{(1)}(n)\rceil$ and terms in GP'_d.

Repeating the above process finitely many time (at most d-times) we get that $g \simeq_d$ finite sum of terms in GP'_d. Thus $g \in GP'_d$ by Lemma 4.12 and Proposition 4.11 (2). The proof is now finished. □

5. Proof of Theorem B(1)

In this section, we will prove Theorem B(1), i.e. we will show that if $A \in \mathcal{F}_{d,0}$ then there are $k \in \mathbb{N}$, $P_i \in GP_d$ $(1 \le i \le k)$ and $\epsilon_i > 0$ such that

$$A \supset \bigcap_{i=1}^{k} \{n \in \mathbb{Z} : P_i(n) \pmod{\mathbb{Z}} \in (-\epsilon_i, \epsilon_i)\}.$$

We remark that by Section 3.2, it is sufficient to consider the case when the group G is a connected, simply-connected d-step nilpotent Lie group.

5.1. **Notations.** Let $X = G/\Gamma$ with G a connected, simply-connected d-step nilpotent Lie group, Γ a uniform subgroup. Let $T : X \longrightarrow X$ be the nilrotation induced by $a \in G$.

We assume $\mathfrak{g}$ is the Lie algebra of G over $\mathbb{R}$, and $\exp : \mathfrak{g} \longrightarrow G$ is the exponential map. Consider

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supsetneq \mathfrak{g}^{(2)} \supsetneq \dots \supsetneq \mathfrak{g}^{(d)} \supsetneq \mathfrak{g}^{(d+1)} = \{0\}.$$

We note that

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}.$$

There is a Mal'cev basis $\mathcal{X} = \{X_1, \dots, X_m\}$ for $\mathfrak{g}$ with

(1) For each $j = 0, \dots, m-1$ the subspace $\eta_j := \mathrm{Span}(X_{j+1}, \dots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_j := \exp \eta_j$ is a normal Lie subgroup of G.
(2) For every $0 < i < d$ we have $G_i = H_{l_{i-1}+1}$.
(3) Each $g \in G$ can be written uniquely as $\exp(t_1X_1)\exp(t_2X_2)\dots\exp(t_mX_m)$, for $t_i \in \mathbb{R}$.
(4) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$,

where $G = G_1$, $G_{i+1} = [G_i, G]$ with $G_{d+1} = \{e\}$. Thus, there are $0 = l_0 < l_1 < \dots < l_{d-1} < l_d = m$ such that $\mathrm{Span} - \{X_{l_i+1}, \dots, X_m\} = \mathfrak{g}^{(i+1)}$ for $i = 0, 1, \dots, d-1$.

Definition 5.1. Define $o(0) = 0$ and $o(i) = j$ if $l_{j-1} + 1 \le i \le l_j, 2 \le j \le d-1$.

5.2. **Some lemmas.** We need several lemmas. Note that if

$$\exp(t_1X_1)\dots\exp(t_mX_m)=\exp(u_1X_1+\dots+u_mX_m)$$

it is known that [10, 26] each t_i is a polynomial of $u_1,\dots,u_m$ and each u_i is a polynomial of $t_1,\dots,t_m$. For our purpose we need to know the precise degree of the polynomials.

Lemma 5.2. *Let $\{X_1,\dots,X_m\}$ be a Mal'cev bases for G/Γ. Assume that*

$$exp(t_1X_1)\dots exp(t_mX_m)=exp(u_1X_1+\dots+u_mX_m).$$

Then we have

(1) *$u_i=t_i$ for $1\le i\le l_1$ and if $l_{j-1}+1\le i\le l_j,\ 2\le j\le d$ then*

$$u_i=t_i+\sum_{\substack{k_1o(1)+\dots+k_mo(m)\le o(i),\\ k_0\le m-2,\ 0\le k_1,\dots,k_m\le m}}c_{k_1,\dots,k_m,i}t_1^{k_1}\dots t_m^{k_m},$$

where k_0 is the number of $0's$ appearing in $\{k_1,\dots,k_m\}$.

(2) *$t_i=u_i$ for $1\le i\le l_1$ and if $l_{j-1}+1\le i\le l_j,\ 2\le j\le d$ then*

$$t_i=u_i+\sum_{\substack{k_1o(1)+\dots+k_mo(m)\le o(i),\\ k_0\le m-2,\ 0\le k_1,\dots,k_m\le m}}d_{k_1,\dots,k_m,i}u_1^{k_1}\dots u_m^{k_m},$$

where k_0 is the number of $0's$ appearing in $\{k_1,\dots,k_m\}$.

Proof. (1). It is easy to see that if $m=1$ then $d=1$ and (1) holds. So we may assume that $m\ge 2$. For $s\in\{0,1,\dots,m\}^{\{1,\dots,m\}}$, let $\{i_1<\dots<i_n\}$ be the collection of $p's$ with $s(p)\neq 0$. Let

$$X_s=[X_{s(i_1)},[X_{s(i_2)},\dots,[X_{s(i_{n-1})},X_{s(i_n)}]]].$$

For each $0\le p\le m$ let $k_p(s)$ be the number of $p's$ appearing in s (as usual, the cardinality of the empty set is defined as 0). Using the CBH formula $m-1$ times and the condition $\mathfrak{g}^{(d+1)}=\{0\}$ it is easy to see that $(t_1X_1)*\dots*(t_mX_m)$ is the sum of $\sum_{i=1}^m t_iX_i$ and the terms

$$constant\times t_{q_1}\dots t_{q_n}[X_{q_1},[X_{q_2},\dots,[X_{q_{n-1}},X_{q_n}]]],\ m\ge n\ge 2,$$

i.e. $\exp(t_1X_1)\dots\exp(t_mX_m)$ can be written as

$$\exp(\sum_{j=1}^m t_jX_j+\sum_{\substack{s\in\{0,1,\dots,m\}^{\{1,\dots,m\}}\\ k_0(s)\le m-2}}c'_st_1^{k_1(s)}\dots t_m^{k_m(s)}X_s).$$

Note that $X_s\subset\mathfrak{g}^{(\sum_{j=1}^m k_j(s)o(j))}$. Let $X_s=\sum_{j=1}^m c'_{s,j}X_j$. Thus, $c'_{s,1},\dots,c'_{s,i}=0$ if $\sum_{j=1}^m k_j(s)o(j)>o(i)$. Thus, $u_i=t_i$ for $1\le i\le l_1$ and if $l_{j-1}+1\le i\le l_j,\ 2\le j\le d$ then the coefficient of X_i is

$$u_i=t_i+\sum_{\substack{k_1o(1)+\dots+k_mo(m)\le o(i),\\ k_0\le m-2,\ 0\le k_1,\dots,k_m\le m}}c_{k_1,\dots,k_m,i}t_1^{k_1}\dots t_m^{k_m}.$$

Note that when $k_1o(1)+\dots+k_mo(m)\le o(i)$ and $k_0\le m-2$, we have that $k_i=k_{i+1}=\dots=k_m=0$ and some other restrictions. For example, when $l_1+1\le i\le l_2$,

$t_1^{k_1}\dots t_m^{k_m}=t_{i_1}t_{i_2}$ with $1\le i_1,i_2\le l_1$; and when $l_2+1\le i\le l_3$, $t_1^{k_1}\dots t_m^{k_m}=t_{i_1}t_{i_2}t_{i_3}$ with $1\le i_1,i_2,i_3\le l_1$ or $t_{i_1}t_{i_2}$ with $1\le i_1\le l_1$ and $l_1+1\le i_2\le l_2$.

(2) It is easy to see that $t_i=u_i$ for $1\le i\le l_1$. If $d=1$ (2) holds, and thus we assume that $d\ge 2$. We show (2) by induction. We assume that

$$t_p=u_p+\sum_{\substack{k'_1o(1)+\ldots+k'_mo(m)\le o(p),\\ k'_0\le m-2,\ 0\le k'_1,\ldots,k'_m\le m}} d_{k'_1,\ldots,k'_m,p}u_1^{k'_1}\dots u_m^{k'_m}, \tag{5.1}$$

for all p with $l_1+1\le p\le i$.

Since

$$u_{i+1}=t_{i+1}+\sum_{\substack{k_1o(1)+\ldots+k_mo(m)\le o(i+1),\\ k_0\le m-2,\ 0\le k_1,\ldots,k_m\le m}} c_{k_1,\ldots,k_m,i+1}t_1^{k_1}\dots t_m^{k_m},$$

we have that

$$t_{i+1}=u_{i+1}-\sum_{\substack{k_1o(1)+\ldots+k_mo(m)\le o(i+1),\\ k_0\le m-2,\ 0\le k_1,\ldots,k_m\le m}} c_{k_1,\ldots,k_m,i+1}t_1^{k_1}\dots t_m^{k_m}.$$

Since $o(i+1)\le o(i)+1$ and $k_0\le m-2$ we have that if $k_1o(1)+\ldots+k_mo(m)\le o(i+1)$ then $k_po(p)\le o(i)$ for each $1\le p\le m$, which implies that $k_{i+1},\ldots,k_m=0$. By the induction each t_p $(1\le p\le i)$ is a polynomial of $u_1,\ldots,u_m$ of degree at most $\sum_{j=1}^m k'_j\le o(p)$(see Equation (5.1)) thus

$$\sum_{\substack{k_1o(1)+\ldots+k_mo(m)\le o(i+1),\\ k_0\le m-2,\ 0\le k_1,\ldots,k_m\le m}} c_{k_1,\ldots,k_m,i+1}t_1^{k_1}\dots t_m^{k_m}$$

is a polynomial of $u_1,\ldots,u_m$ of degree at most $\sum_{p=1}^m k_p\le o(i+1)$. Rearranging the coefficients we get (2). Note that $k_0\le m-2$ is satisfied automatically. $\square$

Lemma 5.3. *Assume that*

$$x=exp(x_1X_1+\cdots+x_mX_m)\quad \textit{and } y=exp(y_1X_1)\dots exp(y_mX_m).$$

Then

$$xy^{-1}=exp(\sum_{i=1}^{l_1}(x_i-y_i)X_i+\sum_{i=l_1+1}^{m}((x_i-y_i)+P_{i,1}(\{y_p\})+P_{i,2}(\{x_p\},\{y_p\}))X_i),$$

where $P_{i,1}(\{y_p\}),P_{i,2}(\{x_p\},\{y_p\})$ are polynomials of degree at most $o(i)$.

Proof. By Lemma 5.2 we have

$$xy^{-1}=\exp(X)\exp(Y)$$

where $X=\sum_{i=1}^m x_iX_i$ and

$$Y=-\sum_{i=1}^m y_iX_i-\sum_{i=l_1+1}^m(\sum_{\substack{k'_1o(1)+\ldots+k'_mo(m)\le o(i),\\ k'_0\le m-2,0\le k'_1,\ldots,k'_m\le m}} c_{k'_1,\ldots,k'_m,i}y_1^{k'_1}\dots y_m^{k'_m})X_i.$$

Using the CBH formula we get that

$$\begin{aligned} xy^{-1} &= \exp(X*Y) = \exp(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\cdots) \\ &= \exp(\sum_{i=1}^{m}(x_i-y_i)X_i + \sum_{i=l_1+1}^{m}(P_{i,1}(\{y_p\})+P_{i,2}(\{x_p\},\{y_p\}))X_i) \\ &= \exp(\sum_{i=1}^{l_1}(x_i-y_i)X_i + \sum_{i=l_1+1}^{m}((x_i-y_i)+P_{i,1}(\{y_p\})+P_{i,2}(\{x_p\},\{y_p\}))X_i) \end{aligned}$$

where

$$P_{i,1}(\{y_p\}) = -\sum_{\substack{k'_1 o(1)+\ldots+k'_m o(m)\le o(i),\\ k'_0\le m-2, 0\le k'_1,\ldots,k'_m\le m}} c_{k'_1,\ldots,k'_m,i}\, y_1^{k'_1}\ldots y_m^{k'_m},$$

and

$$P_{i,2}(\{x_p\},\{y_p\}) = \sum_{\substack{\sum_{j=1}^m (k_j+k'_j)o(j)\le o(i),\\ 0\le k_1,\ldots,k_m,k'_1,\ldots,k'_m\le m\\ k_0\le m-1, k'_0\le m-1}} e_{k_1,\ldots,k_m}^{k'_1,\ldots,k'_m} x_1^{k_1}\ldots x_m^{k_m} y_1^{k'_1}\ldots y_m^{k'_m}.$$

Note that the reason $P_{i,2}$ has the above form follows from the fact that $[\mathfrak{g}^{(i)},\mathfrak{g}^{(j)}]\subset \mathfrak{g}^{(i+j)}$, $\mathfrak{g}^{(d+1)}=\{0\}$ and a discussion similar to the one used in Lemma 5.2. □

5.3. **Proof of Theorem B(1).** Let $X=G/\Gamma$ with G a connected, simply-connected d-step nilpotent Lie group, Γ a uniform subgroup. Let $T: X\longrightarrow X$ be the nilrotation induced by $a\in G$. Assume that $A\supset N(x\Gamma,U)$ with $x\in X$, $x\Gamma\in U$ and $U\subset G/\Gamma$ open. By Proposition 3.4 we may assume that U is an open neighborhood of $e\Gamma$ in G/Γ, where e is the unit element of G, i.e. $A\supset N(e\Gamma,U)$.

Assume that $a=\exp(a_1X_1+\cdots+a_mX_m)$, where $a_1,\cdots a_m\in\mathbb{R}$. Then

$$a^n=\exp(na_1X_1+\ldots+na_mX_m)$$

for any $n\in\mathbb{Z}$. For $h=\exp(h_1X_1)\ldots\exp(h_mX_m)$, where $h_1,\cdots,h_m\in\mathbb{R}$, write

$$a^nh^{-1}=\exp(p_1X_1+\ldots+p_mX_m)=\exp(w_1X_1)\ldots\exp(w_mX_m).$$

Then by Lemma 5.3 we have $p_i=na_i-h_i$ for $1\le i\le l_1$ and if $l_{j-1}+1\le i\le l_j,\ 2\le j\le d$ then

$$p_i = na_i-h_i+P_{i,1}(\{h_p\})+\sum_{\substack{\sum_{p=1}^m (k_p+k'_p)o(p)\le o(i),\\ 0\le k_1,\ldots,k_m,k'_1,\ldots,k'_m\le m\\ k_0\le m-1, k'_0\le m-1}} e_{k_p,k'_p} n^{k_1+\ldots+k_m} h_1^{k'_1}\ldots h_m^{k'_m}, \tag{5.2}$$

where $P_{i,1}(\{h_p\}) = -\sum_{\substack{k'_1 o(1)+\ldots+k'_m o(m)\le o(i),\\ k'_0\le m-2, 0\le k'_1,\ldots,k'_m\le m}} c_{k'_1,\ldots,k'_m,i}\, h_1^{k'_1}\ldots h_m^{k'_m}$ and $e_{k_p,k'_p}=e_{k_1,\ldots,k_m}^{k'_1,\ldots,k'_m}a_1^{k_1}\ldots a_m^{k_m}$.

Changing the exponential coordinates to Mal'sev coordinates (Lemma 5.2), we get that $w_i = na_i - h_i$ for $1 \le i \le l_1$ and if $l_{j-1}+1 \le i \le l_j,\ 2 \le j \le d$ then

$$w_i = p_i + \sum_{\substack{k_1 o(1)+\ldots+k_m o(m)\le o(i),\\ k_0\le m-2,\ 0\le k_1,\ldots,k_m\le m}} d_{k_1,\ldots,k_m,i} p_1^{k_1}\ldots p_m^{k_m},$$

in this case using (5.2) it is not hard to see that w_i is the sum of $-h_i$ and $Q_i = Q_i(n, h_1, \ldots, h_{i-1})$ such that Q_i is the sum of terms

$$c(k, k_1, \ldots, k_{i-1}) n^k h_1^{k_1} \ldots h_{i-1}^{k_{i-1}}$$

with $k + k_1 o(1) + \ldots + k_{i-1} o(i-1) \le o(i)$ (see the argument of Lemma 5.2(2)). Note that if $k = 0$ then $k_0 \le m-2$, and if $k_1 = \ldots = k_m = 0$ then $k \ge 1$.

For a given $n \in \mathbb{Z}$, let $h_i(n) = \lceil na_i \rceil$ if $1 \le i \le l_1$, and let $h_i(n) = \lceil Q_i(n, h_1(n), \ldots, h_{i-1}(n)) \rceil$ if $l_{j-1}+1 \le i \le l_j,\ 2 \le j \le d$. Again a similar argument as in the proof of Lemma 5.2(2) shows that $h_i(n)$ is well defined and is a generalized polynomial of degree of most $o(i) \le d$. For example, if $l_1 + 1 \le i \le l_2$ then

$$p_i = na_i - h_i + \sum_{1\le i_1<i_2\le l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1\le j_1\le l_1} c(j_1, i) n h_{j_1}.$$

So

$$\begin{aligned} w_i &= na_i - h_i + \sum_{1\le i_1<i_2\le l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1\le j_1\le l_1} c(j_1, i) n h_{j_1} + \sum_{1\le i_1<i_2\le l_1} d(i_1, i_2, i) p_{i_1} p_{i_2} \\ &= na_i - h_i + \sum_{1\le i_1<i_2\le l_1} c(i_1, i_2, i) h_{i_1} h_{i_2} + \sum_{1\le j_1\le l_1} c(j_1, i) n h_{j_1} \\ &\qquad + \sum_{1\le i_1<i_2\le l_1} d(i_1, i_2, i)(na_{i_1} - h_{i_1})(na_{i_2} - h_{i_2}). \end{aligned}$$

Thus if we let $h_i(n) = \lceil na_i \rceil$, $1 \le i \le l_1$ then if $l_1 + 1 \le i \le l_2$

$$\begin{aligned} h_i(n) = \lceil na_i &+ \sum_{1\le i_1<i_2\le l_1} c(i_1, i_2, i) \lceil na_{i_1} \rceil \lceil na_{i_2} \rceil + \sum_{1\le j_1\le l_1} c(j_1, i) n \lceil na_{j_1} \rceil \\ &+ \sum_{1\le i_1<i_2\le l_1} d(i_1, i_2, i)(na_{i_1} - \lceil na_{i_1} \rceil)(na_{i_2} - \lceil na_{i_2} \rceil) \rceil. \end{aligned}$$

That is,

$$h_i(n) = \lceil na_i + n^2 a_i' + \sum_{1\le i_1<i_2\le l_1} c'(i_1, i_2, i) \lceil na_{i_1} \rceil \lceil na_{i_2} \rceil + \sum_{1\le j_1\le l_1} c'(j_1, i) n \lceil na_{j_1} \rceil \rceil$$

is a generalized polynomial of degree at most 2 in n.

Next we let $w_i(n) = na_i - h_i(n) = na_i - \lceil na_i \rceil$ for $1 \le i \le l_1$ and if $l_{j-1}+1 \le i \le l_j,\ 2 \le j \le d$, let

$$\begin{aligned} w_i(n) &= Q_i(n, h_1(n), \ldots, h_{i-1}(n)) - h_i(n) \\ &= Q_i(n, h_1(n), \ldots, h_{i-1}(n)) - \lceil Q_i(n, h_1(n), \ldots, h_{i-1}(n)) \rceil. \end{aligned}$$

Since $Q_i(n, h_1(n), \ldots, h_{i-1}(n))$ is the sum of terms

$$c(k, k_1, \ldots, k_{i-1})n^k h_1^{k_1}(n) \ldots h_{i-1}^{k_{i-1}}(n)$$

with $k + k_1 o(1) + \ldots + k_m o(m) \le o(i)$ and $h_i(n)$ is a generalized polynomial of degree of most $o(i) \le d$, we have $w_i(n)$ is a generalized polynomial of degree of most $o(i) \le d$.

Let $h(n) = \exp(h_1(n)X_1) \cdots \exp(h_m(n)X_m)$. Then $h(n) \in \Gamma$ and $a^n h(n)^{-1} = \exp(w_1(n)X_1) \cdots \exp(w_m(n)X_m)$. Choose $0 < \epsilon << \frac{1}{2}$ such that

$$\{g\Gamma : \rho(g\Gamma, e\Gamma) < \epsilon\} \subset U.$$

Then

$$A \supset N(e\Gamma, U) \supset \{n \in \mathbb{Z} : \rho(a^n\Gamma, e\Gamma) < \epsilon\}.$$

We get that (see Definition 3.3)

$$\rho(a^n\Gamma, e\Gamma) \le \rho(a^n h(n)^{-1}, e) \le \max_{1 \le k \le m} \{||w_k(n)||\}.$$

So if $n \in \bigcap_{i=1}^m \{n \in \mathbb{Z} : w_i(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\}$ then $\rho(a^n\Gamma, e\Gamma) < \epsilon$ which implies that $n \in N(e\Gamma, U) \subset A$. That is,

$$A \supset \bigcap_{i=1}^{m} \{n \in \mathbb{Z} : w_i(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\}.$$

This ends the proof of Theorem B(1).

6. Proof of Theorem B(2)

In this section, we aim to prove Theorem B(2), i.e. $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$. To do this first we make some preparations, then derive some results under the inductive assumption, and finally give the proof. Note that in the construction the nilpotent matrix Lie group is used.

More precisely, to show $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ we need only to prove $\mathcal{F}_{d,0} \supset \mathcal{F}_{SGP_d}$ by Theorem B. To do this, for a given $F \in \mathcal{F}_{SGP_d}$ we need to find a d-step nilsystem (X, T), $x_0 \in X$ and a neighborhood U of x_0 such that $F \supset N(x_0, U)$. In the process doing this, we find that it is convenient to consider a finite sum of specially generalized polynomials $P(n; \alpha_1, \ldots, \alpha_r)$ (defined in (6.4)) instead of considering a single specially generalized polynomial. We can prove that $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ if and only if $\{n \in \mathbb{Z} : ||P(n; \alpha_1, \cdots, \alpha_d)|| < \epsilon\} \in \mathcal{F}_{d,0}$ for any $\alpha_1, \cdots, \alpha_d \in \mathbb{R}$ and $\epsilon > 0$ (Theorem 6.7). We choose (X, T) as the closure of the orbit of Γ in $\mathbb{G}_d/\Gamma$ (the nilrotation is induced by a matrix $A \in \mathbb{G}_d$), and consider the most right-corner entry $z_1^d(n)$ in $A^n B_n$ with $B_n \in \Gamma$. We finish the proof by showing that $P(n; \alpha_1, \cdots, \alpha_d) \simeq_d z_1^d(n)$ and $\{n \in \mathbb{Z} : ||z_1^d(n)|| < \epsilon\} \in \mathcal{F}_{d,0}$ for any $\epsilon > 0$.

6.1. **Some preparations.** For a matrix A in $\mathbb{G}_d$ we now give a precise formula of A^n.

Lemma 6.1. *Let* $\mathbf{x} = (x_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$. *For* $n \in \mathbb{N}$, *assume that* $\mathbf{x}(n) = (x_i^k(n))_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ *satisfies* $\mathbf{M}(\mathbf{x}(n)) = \mathbf{M}(\mathbf{x})^n$, *then*

$$x_i^k(n) = \binom{n}{1} P_1(\mathbf{x}; i, k) + \binom{n}{2} P_2(\mathbf{x}; i, k) + \cdots + \binom{n}{k} P_k(\mathbf{x}; i, k) \tag{6.1}$$

for $1 \le k \le d$ *and* $1 \le i \le d-k+1$, *where* $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ *for* $n, k \in \mathbb{N}$ *and*

$$P_\ell(\mathbf{x}; i, k) = \sum_{\substack{(s_1, s_2, \cdots, s_\ell) \in \{1,2,\cdots,k\}^\ell \\ s_1+s_2+\cdots+s_\ell = k}} x_i^{s_1} x_{i+s_1}^{s_2} x_{i+s_1+s_2}^{s_3} \cdots x_{i+s_1+s_2+\cdots+s_{\ell-1}}^{s_\ell}$$

for $1 \le k \le d$, $1 \le i \le d-k+1$ *and* $1 \le \ell \le k$.

Proof. Let $x_i^0 = 1$ and $x_i^0(m) = 1$ for $1 \le i \le d$ and $m \in \mathbb{N}$. By (3.2), it is not hard to see that

$$x_i^k(m+1) = \sum_{j=0}^{k} x_i^{k-j}(m) \cdot x_{i+k-j}^j \tag{6.2}$$

for $1 \le k \le d$, $1 \le i \le d-k+1$ and $m \in \mathbb{N}$.

Now we do induction for k. When $k = 1$, $x_i^1(1) = x_i^1$ and $x_i^1(m+1) = x_i^1(m) + x_i^1$ for $m \in \mathbb{N}$ by (6.2). Hence $x_i^1(n) = n x_i^1 = \binom{n}{1} P_1(\mathbf{x}; i, 1)$. That is, (6.1) holds for each $1 \le i \le d$ and $n \in \mathbb{N}$ if $k = 1$.

Assume that $1 \le \ell \le d-1$, and (6.1) holds for each $1 \le k \le \ell$, $1 \le i \le d-k+1$ and $n \in \mathbb{N}$. For $k = \ell+1$, we make induction on n. When $n = 1$ it is clear

$$x_i^k(1) = x_i^k = \binom{1}{1} P_1(\mathbf{x}; i, k) + \binom{1}{2} P_2(\mathbf{x}; i, k) + \cdots + \binom{1}{k} P_k(\mathbf{x}; i, k)$$

for $1 \le i \le d-k+1$. That is, (6.1) holds for $k = \ell+1$, $1 \le i \le d-k+1$ and $n = 1$. Assume for $n = m \ge 1$, (6.1) holds for $k = \ell+1$, $1 \le i \le d-k+1$ and $n = m$. For $n = m+1$, by (6.2)

$$\begin{aligned}
x_i^k(n) &= x_i^k(m) + \Big(\sum_{j=1}^{k-1} x_i^{k-j}(m) \cdot x_{i+k-j}^j \Big) + x_i^k \\
&= x_i^k(m) + \Big(\sum_{j=1}^{k-1} \big(\sum_{r=1}^{k-j} \binom{m}{r} P_r(\mathbf{x}; i, k-j) \big) \cdot x_{i+k-j}^j) \Big) + x_i^k \\
&= x_i^k(m) + \Big(\sum_{r=1}^{k-1} \big(\sum_{j=1}^{k-r} P_r(\mathbf{x}; i, k-j) x_{i+k-j}^j \big) \binom{m}{r} \Big) + x_i^k \\
&= x_i^k(m) + \Big(\sum_{r=1}^{k-1} \big(\sum_{j=r}^{k-1} P_r(\mathbf{x}; i, j) x_{i+j}^{k-j} \big) \binom{m}{r} \Big) + x_i^k
\end{aligned}$$

for $1 \le i \le d-k+1$. Note that

$$\sum_{j=r}^{k-1} P_r(\mathbf{x};i,j)x_{i+j}^{k-j} = \sum_{j=r}^{k-1} \sum_{\substack{(s_1,\cdots,s_r)\in\{1,2,\cdots,k-1\}^r \\ s_1+\cdots+s_r=j}} x_i^{s_1}x_{i+s_1}^{s_2}\cdots x_{i+s_1+\cdots+s_{r-1}}^{s_r}x_{i+j}^{k-j}$$

which is equal to

$$\sum_{\substack{(s_1,\cdots,s_r,s_{r+1})\in\{1,2,\cdots,k-1\}^{r+1} \\ s_1+s_2+\cdots+s_r+s_{r+1}=k}} x_i^{s_1}x_{i+s_1}^{s_2}\cdots x_{i+s_1+\cdots+s_{r-1}}^{s_r}x_{i+s_1+\cdots+s_{r-1}+s_r}^{s_{r+1}} = P_{r+1}(\mathbf{x};i,k)$$

for $1 \le r \le k-1$ and $1 \le i \le d-k+1$. Collecting terms we have

$$\begin{aligned} x_i^k(n) &= x_i^k(m) + \Big(\sum_{r=1}^{k-1} P_{r+1}(\mathbf{x};i,k)\tbinom{m}{r}\Big) + x_i^k \\ &= x_i^k(m) + \Big(\sum_{r=2}^{k} P_r(\mathbf{x};i,k)\tbinom{m}{r-1}\Big) + P_1(\mathbf{x};i,k) \\ &= \Big(\sum_{r=1}^{m} P_r(\mathbf{x};i,k)\tbinom{m}{r}\Big) + \Big(\sum_{r=2}^{k} P_r(\mathbf{x};i,k)\tbinom{m}{r-1}\Big) + P_1(\mathbf{x};i,k). \end{aligned}$$

Rearranging the order we get

$$\begin{aligned} x_i^k(n) &= (m+1)P_1(\mathbf{x};i,k) + \sum_{r=2}^{k}\Big(\tbinom{m}{r} + \tbinom{m}{r-1}\Big)P_r(\mathbf{x};i,k) \\ &= \sum_{r=1}^{k} \tbinom{m+1}{r}P_r(\mathbf{x};i,k) = \sum_{r=1}^{k} \tbinom{n}{r}P_r(\mathbf{x};i,k) \end{aligned}$$

for $1 \le i \le d-k+1$. This ends the proof of the lemma. □

Remark 6.2. By the above lemma, we have

$$P_1(\mathbf{x};i,k) = x_i^k \text{ and } P_k(\mathbf{x};i,k) = x_i^1x_{i+1}^1\cdots x_{i+k-1}^1$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$.

6.2. **Consequences under the inductive assumption.** We will use induction to show Theorem B(2). To make the proof clearer, we derive some results under the following inductive assumption.

$$\mathcal{F}_{d-1,0} \supset \mathcal{F}_{GP_{d-1}}, \tag{6.3}$$

where $d \in \mathbb{N}$ with $d \ge 2$. For that purpose, we need more notions and lemmas. The proof of Lemma 6.3 is similar to the one of Lemma 4.7, where $\mathcal{W}_d$ is defined in Definition 4.6.

Lemma 6.3. *Under the assumption* (6.3), *one has for any $p(n) \in \mathcal{W}_d$ and $\epsilon > 0$,*

$$\{n \in \mathbb{Z} : p(n) \ (mod\ \mathbb{Z}) \in (-\epsilon,\epsilon)\} \in \mathcal{F}_{d-1,0}.$$

Definition 6.4. For $r \in \mathbb{N}$, we define

$$\widetilde{GP}_r = \{p(n) \in GP_r : \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{r,0} \text{ for any } \epsilon > 0\}.$$

Remark 6.5. It is clear that for $p(n) \in GP_r$, $p(n) \in \widetilde{GP}_r$ if and only if $-p(n) \in \widetilde{GP}_r$. Since $\mathcal{F}_{r,0}$ is a filter, if $p_1(n), p_2(n), \cdots, p_k(n) \in \widetilde{GP}_r$ then

$$p_1(n) + p_2(n) + \cdots + p_k(n) \in \widetilde{GP}_r.$$

Moreover by the definition of $\widetilde{GP}_d$, we know that $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ if and only if $\widetilde{GP}_d = GP_d$.

Lemma 6.6. *Let $p(n), q(n) \in GP_d$ with $p(n) \simeq_d q(n)$. Under the assumption* (6.3), *$p(n) \in \widetilde{GP}_d$ if and only if $q(n) \in \widetilde{GP}_d$.*

Proof. This follows from Lemma 6.3 and the fact that $\mathcal{F}_{d,0}$ is a filter. $\square$

For $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{R}, r \in \mathbb{N}$, we define

$$\begin{aligned} &P(n; \alpha_1, \alpha_2, \cdots, \alpha_r) \\ =&\sum_{\ell=1}^{r} \sum_{\substack{j_1,\cdots,j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = r}} (-1)^{\ell-1} L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1}, \frac{n^{j_2}}{j_2!}\prod_{r_2=1}^{j_2}\alpha_{j_1+r_2}, \cdots, \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1} j_t + r_\ell}\Big) \end{aligned} \tag{6.4}$$

where the definition of L is given in (4.1).

Theorem 6.7. *Under the assumption* (6.3), *the following properties are equivalent:*

(1) $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$.

(2) $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \in \widetilde{GP}_d$ *for any* $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$, *that is*

$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \ (\mathit{mod}\ \mathbb{Z}) \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{d,0}$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$ *and* $\epsilon > 0$.

(3) $SGP_d \subset \widetilde{GP}_d$.

Proof. (1) $\Rightarrow$ (2). Assume $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$. By the definition of $\widetilde{GP}_d$, we know that $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$ if and only if $\widetilde{GP}_d = GP_d$. Particularly $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \in \widetilde{GP}_d$ for any $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$.

(3) $\Rightarrow$ (1). Assume that $\mathrm{SGP}_d \subset \widetilde{GP}_d$. Then $\mathcal{F}_{d,0} \supseteq \mathcal{F}_{SGP_d}$. Moveover $\mathcal{F}_{d,0} \supset \mathcal{F}_{SGP_d} = \mathcal{F}_{GP_d}$ by Theorem 4.13.

(2) $\Rightarrow$ (3). Assume that $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \in \widetilde{GP}_d$ for any $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$. We define

$$\Sigma_d = \{(j_1, j_2, \cdots, j_\ell) : \ell \in \{1, 2, \cdots, d\}, j_1, j_2, \cdots, j_\ell \in \mathbb{N} \text{ and } \sum_{t=1}^{\ell} j_t = d\}.$$

For $(j_1, j_2, \cdots, j_\ell), (r_1, r_2, \cdots, r_s) \in \Sigma_d$, we say $(j_1, j_2, \cdots, j_\ell) > (r_1, r_2, \cdots, r_s)$ if there exists $1 \le t \le \ell$ such that $j_t > r_s$ and $j_i = r_i$ for $i < t$. Clearly $(\Sigma_d, >)$ is a totally ordered set with the maximal element (d) and the minimal element $(1, 1, \cdots, 1)$.

For $\mathbf{j}=(j_1,j_2,\cdots,j_\ell)\in\Sigma_d$, put

$$\mathcal{L}(\mathbf{j})=\{L(n^{j_1}a_1,\cdots,n^{j_\ell}a_\ell):a_1,\cdots,a_\ell\in\mathbb{R}\}.$$

Now, we have

Claim: $\mathcal{L}(\mathbf{s})\subseteq\widetilde{GP}_d$ for each $\mathbf{s}\in\Sigma_d$.

Proof. We do induction for $\mathbf{s}$ under the order $>$. First, consider the case when $\mathbf{s}=(d)$. Given $a_1\in\mathbb{R}$, we take $\alpha_1=1,\alpha_2=2,\cdots\alpha_{d-1}=d-1$ and $\alpha_d=da_1$. Then for any $1\le j_1\le d-1$, $\frac{n^{j_1}}{j_1!}\prod_{t=1}^{j_1}\alpha_t\in\mathbb{Z}$ for $n\in\mathbb{Z}$. Thus

$$P(n;\alpha_1,\alpha_2,\cdots,\alpha_d)=L(\frac{n^d}{d!}\prod_{t=1}^{d}\alpha_t)=L(n^da_1)\ (\mathrm{mod}\ \mathbb{Z})$$

for any $n\in\mathbb{Z}$. Hence $L(n^da_1)\in\widetilde{GP}_d$ since $P(n;\alpha_1,\alpha_2,\cdots,\alpha_d)\in\widetilde{GP}_d$. Since a_1 is arbitrary, we conclude that $\mathcal{L}((d))\subset\widetilde{GP}_d$.

Assume that for any $\mathbf{s}>\mathbf{i}=(i_1,\cdots,i_k)\in\Sigma_d$, we have $\mathcal{L}(\mathbf{s})\subset\widetilde{GP}_d$. Now consider the case when $\mathbf{s}=\mathbf{i}=(i_1,\cdots,i_k)$. There are two cases.

The first case is $k=d$, $i_1=i_2=\cdots=i_d=1$. Given $a_1,a_2,\cdots,a_d\in\mathbb{R}$, by the assumption we have that for any $(j_1,j_2,\cdots,j_\ell)>\mathbf{i}$, $\mathcal{L}((j_1,j_2,\cdots,j_\ell))\subset\widetilde{GP}_d$. Thus

$$\sum_{\ell=1}^{d-1}\sum_{\substack{j_1,\cdots,j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=r}}(-1)^{\ell-1}L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}a_{r_1},\ \frac{n^{j_2}}{j_2!}\prod_{r_2=1}^{j_2}a_{j_1+r_2},\ \cdots,\ \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}a_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)$$

belongs to $\widetilde{GP}_d$ by the Remark 6.5. This implies

$$P(n;a_1,a_2,\cdots,a_d)-L(na_1,na_2,\cdots,na_d)\in\widetilde{GP}_d$$

by (6.4). Combining this with $P(n;a_1,a_2,\cdots,a_d)\in\widetilde{GP}_d$, we have

$$L(na_1,na_2,\cdots,na_d)\in\widetilde{GP}_d$$

by Remark 6.5. Since $a_1,a_2,\cdots,a_d\in\mathbb{R}$ are arbitrary, we get $\mathcal{L}(\mathbf{i})\subset\widetilde{GP}_d$.

The second case is $\mathbf{i}>(1,1,\cdots,1)$. Given $a_1,a_2,\cdots,a_k\in\mathbb{R}$, for $r=1,2,\cdots,k$, we put $\alpha_{\sum_{t=1}^{r-1}i_t+h}=h$ for $1\le h\le i_r-1$ and $\alpha_{\sum_{t=1}^{r-1}i_t+i_r}=i_ra_r$.

By the assumption, for $(j_1,j_2,\cdots,j_\ell)>\mathbf{i}$,

$$L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\ \frac{n^{j_2}}{j_2!}\prod_{r_2=1}^{j_2}\alpha_{j_1+r_2},\ \cdots,\ \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)\in\widetilde{GP}_d.$$

For $(j_1,j_2,\cdots,j_\ell)<\mathbf{i}$, there exists $1\le u\le k$ such that $j_t=i_t$ for $1\le t\le u-1$ and $i_u>j_u$. Then

$$\frac{n^{j_u}}{j_u!}\prod_{r_u=1}^{j_u}\alpha_{\sum_{t=1}^{u-1}j_t+r_u}=n^{j_u}. \tag{6.5}$$

When $u=1$, by (6.5),

$$L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\cdots,\cdots,\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)\in\mathbb{Z}$$

for any $n\in\mathbb{Z}$. Hence

$$L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\cdots,\cdots,\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)\in\widetilde{GP}_d.$$

When $u>1$, write $\beta_v=\prod_{r_v=1}^{j_v}\alpha_{\sum_{t=1}^{v-1}j_t+r_v}$ for $v=1,2,\cdots,\ell$. Then $\beta_u=1$ and

$$\lceil L(n^{j_u}\beta_u,n^{j_{u+1}}\beta_{u+1}\cdots,n^{j_\ell}\beta_\ell)\rceil=L(n^{j_u}\beta_u,n^{j_{u+1}}\beta_{u+1}\cdots,n^{j_\ell}\beta_\ell).$$

Moreover,

$$L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\cdots,\frac{n^{j_u}}{j_u!}\prod_{r_u=1}^{j_u}\alpha_{\sum_{t=1}^{u-1}j_t+r_\ell},\cdots,\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)$$

is equal to

$$L\left(n^{j_1}\beta_1,\cdots,n^{j_u}\beta_u,\cdots,n^{j_\ell}\beta_\ell\right)=L\left(n^{j_1}\beta_1,\cdots,n^{j_{u-1}}\beta_{u-1}\lceil L(n^{j_u}\beta_u,\cdots,n^{j_\ell}\beta_\ell)\rceil\right)$$

which is equal to

$$\begin{aligned}&L\left(n^{j_1}\beta_1,\cdots,n^{j_{u-1}}\beta_{u-1}L(n^{j_u}\beta_u,n^{j_{u+1}}\beta_{u+1}\cdots,n^{j_\ell}\beta_\ell)\right)\\&=L\left(n^{j_1}\beta_1,\cdots,n^{j_{u-1}+j_u}\beta_{u-1}\beta_u\lceil L(n^{j_{u+1}}\beta_{u+1}\cdots,n^{j_\ell}\beta_\ell)\rceil\right)\\&=L\left(n^{j_1}\beta_1,\cdots,n^{j_{u-1}+j_u}\beta_{u-1}\beta_u,n^{j_{u+1}}\beta_{u+1}\cdots,n^{j_\ell}\beta_\ell\right)\in\widetilde{GP}_d\end{aligned}$$

since $(j_1,\cdots,j_{u-2},j_{u-1}+j_u,j_{u+1},\cdots,j_\ell)>\mathbf{i}$.

Summing up for any $\mathbf{j}=(j_1,\cdots,j_\ell)\in\Sigma_d$ with $\mathbf{j}\neq\mathbf{i}$, we have

$$L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\cdots,\frac{n^{j_u}}{j_u!}\prod_{r_u=1}^{j_u}\alpha_{\sum_{t=1}^{u-1}j_t+r_\ell},\cdots,\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big)\in\widetilde{GP}_d.$$

Combining this with $P(n;\alpha_1,\cdots,\alpha_d)\in\widetilde{GP}_d$, we have

$$\begin{aligned}&L\Big(n^{i_1}a_1,\,n^{i_2}a_2,\,\cdots,\,n^{i_k}a_k\Big)\\&=L\Big(\frac{n^{i_1}}{i_1!}\prod_{r_1=1}^{i_1}\alpha_{r_1},\,\frac{n^{i_2}}{i_2!}\prod_{r_2=1}^{i_2}\alpha_{i_1+r_2},\cdots,\frac{n^{i_k}}{i_k!}\prod_{r_k=1}^{i_k}\alpha_{\sum_{t=1}^{k-1}i_t+r_k}\Big)\in\widetilde{GP}_d\end{aligned}$$

by (6.4) and Remark (6.5). Since $a_1,\cdots,a_k\in\mathbb{R}$ are arbitrary, $\mathcal{L}(\mathbf{i})\subset\widetilde{GP}_d$. □

Finally, since $\mathrm{SGP}_d=\bigcup_{\mathbf{j}\in\Sigma_d}\mathcal{L}(\mathbf{j})$, we have $\mathrm{SGP}_d\subset\widetilde{GP}_d$ by the above Claim. □

6.3. **Proof of Theorem B(2).** We are now ready to give the proof of the Theorem B(2). As we said before, we will use induction to show Theorem B(2). Firstly, for $d = 1$, since $\mathcal{F}_{GP_1} = \mathcal{F}_{SGP_1}$ and $\mathcal{F}_{1,0}$ is a filter, it is sufficient to show for any $a \in \mathbb{R}$ and $\epsilon > 0$,

$$\{n \in \mathbb{Z} : an \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{1,0}.$$

This is obvious since the rotation on the unit circle is a 1-step nilsystem.

Now we assume that $\mathcal{F}_{d-1,0} \supset \mathcal{F}_{GP_{d-1}}$, i.e. the the assumption (6.3) holds. By Theorem 6.7, to show $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$, it remains to prove that $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \in \widetilde{GP}_d$ for any $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{R}$, that is

$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{F}_{d,0}$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$ and $\epsilon > 0$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$ and choose $\mathbf{x} = (x_i^k)_{1 \le k \le d, 1 \le i \le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ with $x_i^1 = \alpha_i$ for $i = 1, 2, \cdots, d$ and $x_i^k = 0$ for $2 \le k \le d$ and $1 \le i \le d-k+1$. Then

$$A = \mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & \alpha_1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \alpha_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & \alpha_{d-1} & 0 \\ 0 & 0 & 0 & \ldots & 1 & \alpha_d \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

For $n \in \mathbb{N}$, if $\mathbf{x}(n) = (x_i^k(n))_{1 \le k \le d, 1 \le i \le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ satisfies $\mathbf{M}(\mathbf{x}(n)) = A^n$, then by Lemma 6.1 and Remark 6.2,

$$x_i^k(n) = \binom{n}{k} P_k(\mathbf{x}; i, k) = \binom{n}{k} x_i^1 x_{i+1}^1 \cdots x_{i+k-1}^1 = \binom{n}{k} \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} \tag{6.6}$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$.

Now we define $f_i^1(n) = \lceil x_i^1(n) \rceil = \lceil n\alpha_i \rceil$ for $1 \le i \le d$ and inductively for $k = 2, 3, \cdots, d$ define

$$f_i^k(n) = \left\lceil x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n) \right\rceil \tag{6.7}$$

for $1 \le i \le d-k+1$. Then we define

$$z_i^1(n) = x_i^1(n) - f_i^1(n)$$

for $1 \le i \le d$ and inductively for $k = 2, 3, \cdots, d$ define

$$z_i^k(n) = x_i^k(n) - \Big(\sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n) \Big) - f_i^k(n) \tag{6.8}$$

for $1 \le i \le d-k+1$.

It is clear that $z_i^k(n) \in GP_k$ for $1 \le k \le d$ and $1 \le i \le d-k+1$. First, we have

Claim: $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \simeq_d z_1^d(n)$.

Since the proof of the Claim is long, the readers find the proof in the following subsection. Now we are going to show $z_1^d(n) \in \widetilde{GP}_d$.

Let $X = \mathbb{G}_d/\Gamma$ be endowed with the metric ρ in Lemma 3.7 and T be the nil-rotation induced by $A \in \mathbb{G}_d$, i.e. $B\Gamma \mapsto AB\Gamma$ for $B \in \mathbb{G}_d$. Since $\mathbb{G}_d$ is a d-step nilpotent Lie group and Γ is a uniform subgroup of $\mathbb{G}_d$, (X, T) is a d-step nilsystem. Let $x_0 = \Gamma \in X$ and Z be the closure of the orbit $\mathrm{orb}(x_0, T)$ of e in X. Then (Z, T) is a minimal d-step nilsystem. We consider ρ as a metric on Z.

For a given $\eta > 0$ choose $\delta > 0$ such that $e^{\delta+\delta^2+\cdots+\delta^d} - 1 < \min\{\frac{1}{2}, \eta\}$. Put

$$U = \{z \in Z : \rho(z, x_0) < \delta\}$$

and

$$S = \{n \in \mathbb{N} : \rho(A^n\Gamma, \Gamma) < \delta\} = \{n \in \mathbb{Z} : T^n x_0 \in U\}.$$

Then $S \in \mathcal{F}_{d,0}$ since (Z, T) is a minimal d-step nilsystem. In the following we are going to show that

$$\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\} \supset S.$$

This clearly implies that $\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\} \in \mathcal{F}_{d,0}$ since $S \in \mathcal{F}_{d,0}$. As $\eta > 0$ is arbitrary, we conclude that $z_1^d(n) \in \widetilde{GP}_d$.

Given $n \in S$, one has $\rho(A^n\Gamma, \Gamma) < \delta$. Since ρ is right-invariant and Γ is a group, there exists $B_n^{-1} \in \Gamma$ such that $\rho(A^n, B_n^{-1}) < \delta$. Take $\mathbf{h}(n) = (-h_i^k(n))_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{Z}^{d(d+1)/2}$ with $\mathbf{M}(\mathbf{h}(n)) = B_n$. By (3.3),

$$\|A^n B_n - I\| \le e^{\delta+\delta^2+\cdots+\delta^d} - 1 < \min\left\{\frac{1}{2}, \eta\right\}. \tag{6.9}$$

Let $\mathbf{y}(n) = (y_i^k(n))_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ such that

$$\mathbf{M}(\mathbf{y}(n)) = A^n B_n = \mathbf{M}(\mathbf{x}(n))\mathbf{M}(\mathbf{h}(n)).$$

By (3.2)

$$y_i^k(n) = x_i^k(n) - \Big(\sum_{j=1}^{k-1} x_i^{k-j}(n) h_{i+k-j}^j(n)\Big) - h_i^k(n) \tag{6.10}$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$. Thus

$$|y_i^k(n)| < \min\{\frac{1}{2}, \eta\} \tag{6.11}$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$ by (6.9). Hence $h_i^1(n) = \lceil x_i^1(n) \rceil = \lceil n\alpha_i \rceil$ for $1 \le i \le d$ and

$$h_i^k(n) = \left\lceil x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) h_{i+k-j}^j(n) \right\rceil \tag{6.12}$$

for $2 \le k \le d$ and $1 \le i \le d-k+1$.

Since $h_i^1(n) = \lceil n\alpha_i \rceil = f_i^1(n)$ for $1 \le i \le d$, one has $h_i^k(n) = f_i^k(n)$ for $2 \le k \le d$ and $1 \le i \le d-k+1$ by (6.7) and (6.12). Moreover by (6.8) and (6.10), we know $z_i^k(n) = y_i^k(n)$ for $2 \le k \le d$ and $1 \le i \le d-k+1$. Combining this with (6.11), $|z_i^k(n)| < \min\{\frac{1}{2}, \eta\}$ for $1 \le k \le d$ and $1 \le i \le d-k+1$. Particularly, $|z_1^d(n)| < \eta$. Thus

$$n \in \{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta, \eta)\},$$

which implies that $\{m \in \mathbb{Z} : z_1^d(m) \pmod{\mathbb{Z}} \in (-\eta,\eta)\} \supset S$. That is, $z_1^d(n) \in \widetilde{GP}_d$.

Finally using the Claim and the fact that $z_1^d(n) \in \widetilde{GP}_d$ we have $P(n;\alpha_1,\alpha_2,\cdots,\alpha_d) \in \widetilde{GP}_d$ by Lemma 6.6. This ends the proof, i.e. we have proved $\mathcal{F}_{d,0} \supset \mathcal{F}_{GP_d}$.

6.4. **Proof of the Claim.** Let

$$u_i^k(n) = z_i^k(n) + f_i^k(n) = x_i^k(n) - \sum_{j=1}^{k-1} x_i^{k-j}(n) f_{i+k-j}^j(n)$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$. Then

$$f_i^k(n) = \lceil u_i^k(n) \rceil$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$.

We define $U(n;j_1) = \frac{n^{j_1}}{j_1!}\prod_{r=1}^{j_1}\alpha_r$ for $1 \le j_1 \le d$. Then inductively for $\ell = 2,3,\cdots,d$ we define

$$\begin{aligned} U(n;j_1,j_2,\cdots,j_\ell) &= (U(n;j_1,\cdots,j_{\ell-1}) - \lceil U(n;j_1,\cdots,j_{\ell-1})\rceil)\frac{n^{j_\ell}}{j_\ell!}\prod_{r=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r} \\ &= (U(n;j_1,\cdots,j_{\ell-1}) - \lceil U(n;j_1,\cdots,j_{\ell-1})\rceil)L(\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}) \end{aligned}$$

for $j_1,j_2,\cdots,j_\ell \ge 1$ and $j_1+\cdots+j_\ell \le d$ (see (4.1) for the definition of L).

Next, $U(n;d) = \frac{n^d}{d!}\prod\limits_{r=1}^{d}\alpha_r = L(\frac{n^d}{d!}\prod\limits_{r=1}^{d}\alpha_r)$ and for $2 \le \ell \le d$, $j_1,j_2,\cdots,j_\ell \in \mathbb{N}$ with $j_1+j_2+\cdots+j_\ell = d$, by Lemma 4.9(1)

$$\begin{aligned} U(n;j_1,j_2,\cdots,j_\ell) &= (U(n;j_1,\cdots,j_{\ell-1}) - \lceil U(n;j_1,\cdots,j_{\ell-1})\rceil)L(\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}) \\ &\simeq_d U(n;j_1,\cdots,j_{\ell-1})\lceil L(\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell})\rceil \end{aligned}$$

which is

$$\begin{aligned} &= (U(n;j_1,\cdots,j_{\ell-2}) - \lceil U(n;j_1,\cdots,j_{\ell-2})\rceil)\times \\ &\qquad L(\frac{n^{j_{\ell-1}}}{j_{\ell-1}!}\prod_{r_{\ell-1}=1}^{j_{\ell-1}}\alpha_{\sum_{t=1}^{\ell-2}j_t+r_{\ell-1}}, \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}) \\ &\simeq_d U(n;j_1,\cdots,j_{\ell-2})\lceil L(\frac{n^{j_{\ell-1}}}{j_{\ell-1}!}\prod_{r_{\ell-1}=1}^{j_{\ell-1}}\alpha_{\sum_{t=1}^{\ell-2}j_t+r_{\ell-1}}, \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell})\rceil. \end{aligned}$$

Continuing the above argument we have

$$U(n;j_1,j_2,\cdots,j_\ell) \simeq_d L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1}, \frac{n^{j_2}}{j_2!}\prod_{r_2=1}^{j_2}\alpha_{j_1+r_2}, \cdots, \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell}\Big).$$

That is, for $1 \le \ell \le d$, $j_1, j_2, \cdots, j_\ell \in \mathbb{N}$ with $j_1 + j_2 + \cdots + j_\ell = d$,

(6.13)
$$U(n; j_1, j_2, \cdots, j_\ell) \simeq_d L\Big(\frac{n^{j_1}}{j_1!}\prod_{r_1=1}^{j_1}\alpha_{r_1},\ \frac{n^{j_2}}{j_2!}\prod_{r_2=1}^{j_2}\alpha_{j_1+r_2},\ \cdots,\ \frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1} j_t + r_\ell}\Big).$$

Thus using (6.13) we have

$$P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \simeq_d \sum_{\ell=1}^{d} \sum_{\substack{j_1,\cdots j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = d}} (-1)^{\ell-1} U(n; j_1, j_2, \cdots, j_\ell). \tag{6.14}$$

Next using Lemma 4.9(1), for any $j_1, j_2, \cdots, j_\ell \in \mathbb{N}$ with $j_1 + j_2, \cdots + j_\ell \le d-1$, we have $U(n; j_1, \cdots, j_\ell) f^{d-\sum_{t=1}^{\ell} j_t}_{1+\sum_{t=1}^{\ell} j_t}(n)$ is equal to

$$\begin{aligned}
&U(n; j_1, \cdots, j_\ell)\lceil u^{d-\sum_{t=1}^{\ell} j_t}_{1+\sum_{t=1}^{\ell} j_t}(n)\rceil \\
&\simeq_d \Big(U(n; j_1, \cdots, j_\ell) - \lceil U(n; j_1, \cdots, j_\ell)\rceil\Big) u^{d-\sum_{t=1}^{\ell} j_t}_{1+\sum_{t=1}^{\ell} j_t}(n) \\
&= \Big(U(n; j_1, \cdots, j_\ell) - \lceil U(n; j_1, \cdots, j_\ell)\rceil\Big) \times \\
&\qquad \Big(x^{d-\sum_{t=1}^{\ell} j_t}_{1+\sum_{t=1}^{\ell} j_t}(n) - \sum_{j_{\ell+1}=1}^{d-(\sum_{t=1}^{\ell} j_t)-1} x^{j_{\ell+1}}_{1+\sum_{t=1}^{\ell} j_t}(n) f^{d-\sum_{t=1}^{\ell+1} j_t}_{1+\sum_{t=1}^{\ell+1} j_t}(n)\Big)
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\Big(U(n; j_1, j_2, \cdots, j_\ell) - \lceil U(n; j_1, j_2, \cdots, j_\ell)\rceil\Big) \times \\
&\Bigg(\binom{n}{d-\sum\limits_{t=1}^{\ell} j_t} \prod_{r_{\ell+1}=1}^{d-\sum\limits_{t=1}^{\ell} j_t} \alpha_{\sum\limits_{t=1}^{\ell} j_t + r_{\ell+1}} - \sum_{j_{\ell+1}=1}^{d-\sum\limits_{t=1}^{\ell+1} j_t - 1} \binom{n}{j_{\ell+1}} \prod_{r_{\ell+1}=1}^{j_{\ell+1}} \alpha_{\sum\limits_{t=1}^{\ell} j_t + r_{\ell+1}} f^{d-\sum\limits_{t=1}^{\ell+1} j_t}_{1+\sum\limits_{t=1}^{\ell+1} j_t}(n)\Bigg)
\end{aligned}$$

which is

$$\begin{aligned}
&\simeq_d \Big(U(n; j_1, j_2, \cdots, j_\ell) - \lceil U(n; j_1, j_2, \cdots, j_\ell)\rceil\Big) \times \\
&\Bigg(\frac{n^{d-\sum\limits_{t=1}^{\ell} j_t}}{(d-\sum\limits_{t=1}^{\ell} j_t)!} \prod_{r_{\ell+1}=1}^{d-\sum\limits_{t=1}^{\ell} j_t} \alpha_{\sum\limits_{t=1}^{\ell} j_t + r_{\ell+1}} - \sum_{j_{\ell+1}=1}^{d-\sum\limits_{t=1}^{\ell+1} j_t - 1} \frac{n^{j_{\ell+1}}}{j_{\ell+1}!} \prod_{r_{\ell+1}=1}^{j_{\ell+1}} \alpha_{\sum\limits_{t=1}^{\ell} j_t + r_{\ell+1}} f^{d-\sum\limits_{t=1}^{\ell+1} j_t}_{1+\sum\limits_{t=1}^{\ell+1} j_t}(n)\Bigg) \\
&= U(n; j_1, \cdots, j_\ell, d - \sum_{t=1}^{\ell} j_t) - \sum_{j_{\ell+1}=1}^{d-\sum\limits_{t=1}^{\ell+1} j_t - 1} U(n; j_1, \cdots, j_\ell, j_{\ell+1}) f^{d-\sum\limits_{t=1}^{\ell+1} j_t}_{1+\sum\limits_{t=1}^{\ell+1} j_t}(n).
\end{aligned}$$

Using the fact and Lemma 4.9(1), we have

$$\begin{aligned}
z_1^d(n) &\simeq_d u_1^d(n) = x_1^d(n) - \sum_{j_1=1}^{d-1} x_1^{j_1}(n) f_{1+j_1}^{d-j_1}(n)\\
&= \tbinom{n}{d}\alpha_1\alpha_2\cdots\alpha_d - \sum_{j_1=1}^{d-1}\tbinom{n}{j_1}\alpha_1\alpha_2\cdots\alpha_{j_1} f_{1+j_1}^{d-j_1}(n)\\
&\simeq_d U(n;d) - \sum_{j_1=1}^{d-1} U(n;j_1) f_{1+j_1}^{d-j_1}(n)\\
&\simeq_d U(n;d) - \Big(\sum_{j_1=1}^{d-1}(U(n;j_1,d-j_1) - \sum_{j_2=1}^{d-j_1-1} U(n;j_1,j_2) f_{1+j_1+j_2}^{d-(j_1+j_2)}(n))\Big).
\end{aligned}$$

Continuing this argument we obtain

$$z_1^d(n) \simeq_d \sum_{\ell=1}^{d} \sum_{\substack{j_1,\cdots,j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell}} (-1)^{\ell-1} U(n;j_1,\cdots,j_\ell).$$

Combining this with (6.14), we have proved the Claim.

7. Proof of Theorem C

In this section we will prove Theorem C. That is, we will show that for $d\in\mathbb{N}$ and $F\in\mathcal{F}_{GP_d}$, there exist a minimal d-step nilsystem (X,T) and a nonempty open set U such that

$$F \supset \{n\in\mathbb{Z} : U\cap T^{-n}U\cap\ldots\cap T^{-dn}U\neq\emptyset\}.$$

Let us explain the idea of the proof of Theorem C. Put $\mathcal{N}_d=\{B\subseteq\mathbb{Z}$: there are a minimal d-step nilsystem (X,T) and an open non-empty set U of X with $B\supset\{n\in\mathbb{Z}:\bigcap_{i=0}^d T^{-in}U\neq\emptyset\}\}$. Similar to the proof of Theorem B(2) we first show that $\mathcal{F}_{GP_d}\subseteq\mathcal{N}_d$ if and only if $\{n\in\mathbb{Z}:||P(n;\alpha_1,\cdots,\alpha_d)||<\epsilon\}\in\mathcal{N}_d$ for any $\alpha_1,\cdots,\alpha_d\in\mathbb{R}$ and $\epsilon>0$. We choose (X,T) as the closure of the orbit of Γ in $\mathbb{G}_d/\Gamma$ (the nilrotation is induced by a matrix $A\in\mathbb{G}_d$), define $U\subset X$ depending on a given $\epsilon>0$, put $S=\{n\in\mathbb{Z}:\bigcap_{i=0}^d T^{-in}U\neq\emptyset\}$; and consider the most right-corner entry $z_1^d(m)$ in $A^{nm}BC_m$ with $B\in\mathbb{G}_d$ and $C_m\in\Gamma$ for a given $n\in S$ with $1\le m\le d$. We finish the proof by showing $S\subset\{n\in\mathbb{Z}:||P(n;\alpha_1,\cdots,\alpha_d)||<\epsilon\}$ which implies that $\{n\in\mathbb{Z}:||P(n;\alpha_1,\cdots,\alpha_d)||<\epsilon\}\in\mathcal{N}_d$.

7.1. **The ordinary polynomial case.** To illustrate the idea of the proof of Theorem C, we first consider the situation when the generalized polynomials are the ordinary ones. That is, we want to explain if $p(n)$ is a polynomial of degree d with $p(0)=0$ and $\epsilon>0$, how we can find a d-step nilsystem (X,T), and a nonempty open set $U\subset X$ such that

$$\{n\in\mathbb{Z}:p(n)\ (\mathrm{mod}\ \mathbb{Z})\in(-\epsilon,\epsilon)\}\supset\{n\in\mathbb{Z}:U\cap T^{-n}U\cap\ldots\cap T^{-dn}U\neq\emptyset\}. \tag{7.1}$$

To do this define $T_{\alpha,d} : \mathbb{T}^d \longrightarrow \mathbb{T}^d$ by

$$T_{\alpha,d}(\theta_1, \theta_2, \ldots, \theta_d) = (\theta_1 + \alpha, \theta_2 + \theta_1, \theta_3 + \theta_2, \ldots, \theta_d + \theta_{d-1}),$$

where $\alpha \in \mathbb{R}$. A simple computation yields that

$$T^n_{\alpha,d}(\theta_1, \ldots, \theta_d) = (\theta_1 + n\alpha, n\theta_1 + \theta_2 + \frac{1}{2}n(n-1)\alpha, \ldots, \sum_{i=0}^{d} \tbinom{n}{d-i}\theta_i), \tag{7.2}$$

where $\theta_0 = \alpha$, $n \in \mathbb{Z}$ and $\binom{n}{0} = 1$, $\binom{n}{i} := \frac{\prod_{j=0}^{i-1}(n-j)}{i!}$ for $i = 1, 2, \cdots, d$.

We now prove (7.1) by induction. The case when $d = 1$ is easy, and we assume that for each polynomial of degree $\leq d-1$ (7.1) holds. Now let $p(n) = \sum_{i=1}^{d} \alpha_i n^i$ with $\alpha_i \in \mathbb{R}$. By induction for each $1 \leq i \leq d-1$ there is an i-step nilsystem (X_i, T_i) and an open non-empty subset U_i of X_i such that

$$\{n \in \mathbb{Z} : \alpha_i n^i \pmod{\mathbb{Z}} \in (-\frac{\epsilon}{d}, \frac{\epsilon}{d})\} \supset \{n \in \mathbb{Z} : U_i \cap T_i^{-n} U_i \cap \ldots \cap T_i^{-dn} U_i \neq \emptyset\}.$$

By the Vandermonde's formula, we know

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & d \\ 1 & 2^2 & 3^2 & \ldots & d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{d-1} & 3^{d-1} & \ldots & d^{d-1} \\ 1 & 2^d & 3^d & \ldots & d^d \end{pmatrix}$$

is a non-singular matrix. Hence there are integers $\lambda_1, \lambda_2, \ldots, \lambda_d$ and $\lambda \in \mathbb{N}$ such that the following equation holds:

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & d \\ 1 & 2^2 & 3^2 & \ldots & d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{d-1} & 3^{d-1} & \ldots & d^{d-1} \\ 1 & 2^d & 3^d & \ldots & d^d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{d-1} \\ \lambda_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda \end{pmatrix}$$

That is,

$$\begin{aligned} &\sum_{m=1}^{d} \lambda_m m^j = \lambda_1 + \lambda_2 2^j + \ldots + \lambda_d d^j = 0, \ 1 \leq j \leq d-1; \\ &\sum_{m=1}^{d} \lambda_m m^d = \lambda_1 + \lambda_2 2^d + \ldots + \lambda_d d^d = \lambda. \end{aligned} \tag{7.3}$$

Now let $T_d = T_{\frac{\alpha_d}{\lambda}, d}$ and $Y_d = \mathbb{T}^d$. Let $K_d = d! \sum_{i=1}^{d} |\lambda_i|$, $\epsilon_1 > 0$ with $K_d \epsilon_1 < \epsilon/d$ and $U_d = (-\epsilon_1, \epsilon_1)^d$.

It is easy to see that if $n \in \{n \in \mathbb{Z} : U_d \cap T_d^{-n} U_d \cap \ldots \cap T_d^{-dn} U_d \neq \emptyset\}$ then we know that there is $(\theta_1, \ldots, \theta_d) \in U_d$ such that $T_d^{in}(\theta_1, \ldots, \theta_d) \in U_d$ for each $1 \leq i \leq d$.

Thus, by (7.2) considering the last coordinate we ge that

$$\begin{array}{r}\binom{n}{d}\theta_0+\binom{n}{d-1}\theta_1+\ldots+\binom{n}{0}\theta_d \ (\mathrm{mod}\ \mathbb{Z})\in(-\epsilon_1,\epsilon_1)\\ \binom{2n}{d}\theta_0+\binom{2n}{d-1}\theta_1+\ldots+\binom{2n}{0}\theta_d \ (\mathrm{mod}\ \mathbb{Z})\in(-\epsilon_1,\epsilon_1)\\ \ldots\ldots\ldots\ldots \quad \ldots\\ \binom{dn}{d}\theta_0+\binom{dn}{d-1}\theta_1+\ldots+\binom{dn}{0}\theta_d \ (\mathrm{mod}\ \mathbb{Z})\in(-\epsilon_1,\epsilon_1),\end{array}$$

where $\theta_0=\frac{\alpha_d}{\lambda}$. Multiplying $\binom{in}{d}\theta_0+\binom{in}{d-1}\theta_1+\ldots+\binom{in}{0}\theta_d$ by $\lambda_i d!$ and summing over $i=1,\ldots,d$ we get that

$$\sum_{j=1}^{d}\lambda_j d!\sum_{i=0}^{d}\binom{jn}{d-i}\theta_i=\alpha_d n^d \ (\mathrm{mod}\ \mathbb{Z})\in(-K_d\epsilon_1,K_d\epsilon_1)\subset(-\epsilon/d,\epsilon/d).$$

Choose $x_i\in U_i$ for $1\le i\le d$. Let $x=(x_1,x_2,\ldots,x_d)\in X_1\times\ldots\times X_d$ and X be the orbit x under $T=T_1\times T_2\ldots\times T_d$. Then (X,T) is a d-step nilsystem. If we let $U=(U_1\times U_2\times\ldots\times U_d)\cap X$, then we have (7.1).

By the property of nilsystems and the discussion above it is easy to see

Remark 7.1. Let $k\in\mathbb{N}$, $q_i(x)$ be a polynomial of degree d with $q_i(0)=0$ and $\epsilon_i>0$ for $1\le i\le k$. Then there are a d-step nilsystem (X,T,μ) and $B\subset X$ with $\mu(B)>0$ such that

$$\bigcap_{i=1}^{k}\{n\in\mathbb{Z}:||q_i(n)||<\epsilon_i\}\supset\{n\in\mathbb{Z}:\mu(B\cap T^{-n}B\cap\ldots\cap T^{-dn}B)>0\}$$

7.2. **Some preparation.** For $d\in\mathbb{N}$, define

$$\begin{aligned}\mathcal{N}_d \;=\; &\{B\subseteq\mathbb{Z}:\text{there are a minimal } d\text{-step nilsystem } (X,T) \text{ and an open}\\ &\text{non-empty set } U \text{ of } X \text{ with } B\supset\{n\in\mathbb{Z}:U\cap T^{-n}U\cap\ldots\cap T^{-dn}U\neq\emptyset\}.\}\end{aligned}$$

Hence Theorem C is equivalent to

$$\mathcal{F}_{GP_d}\subseteq\mathcal{N}_d.$$

Lemma 7.2. *For each $d\in\mathbb{N}$, $\mathcal{N}_d$ is a filter.*

Proof. Let $B_1,B_2\in\mathcal{N}_d$. To show $\mathcal{N}_d$ is a filter, it suffices to show $B_1\cap B_2\in\mathcal{N}_d$. By definition, there exist minimal d-step nilsystems $(X_i,T_i), i=1,2$, and a nonempty open set U_i such that

$$B_i\supset\{n\in\mathbb{Z}:U_i\cap T_i^{-n}U_i\cap\ldots\cap T_i^{-dn}U_i\neq\emptyset\}.$$

Taking any minimal point $x=(x_1,x_2)\in X_1\times X_2$, let $X=\overline{\mathrm{orb}(x,T)}$, where $T=T_1\times T_2$. Note that (X,T) is also a minimal d-step nilsystem.

Since $(X_i,T_i), i=1,2$, are minimal, there are $k_i\in\mathbb{N}$ such that $x_i\in T_i^{-k_i}U_i$, $i=1,2$. Let $U=(T_1^{-k_1}U_1\times T_2^{-k_2}U_2)\cap X$, then U is an open set of X. Note that

$$\begin{aligned}&\{n\in\mathbb{Z}:U\cap T^{-n}U\cap\ldots\cap T^{-dn}U\neq\emptyset\}\\ =&\bigcap_{i=1,2}\{n\in\mathbb{Z}:T_i^{-k_i}U_i\cap T_i^{-(k_i+n)}U_i\cap\ldots\cap T_i^{-(k_i+dn)}U_i\neq\emptyset\}\\ =&\bigcap_{i=1,2}\{n\in\mathbb{Z}:U_i\cap T_i^{-n}U_i\cap\ldots\cap T_i^{-dn}U_i\neq\emptyset\}\end{aligned}$$

Hence

$$B_1 \cap B_2 \supset \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}.$$

That is, $B_1 \cap B_2 \in \mathcal{N}_d$ and $\mathcal{N}_d$ is a filter. □

Definition 7.3. For $r \in \mathbb{N}$, define

$$\widehat{GP}_r = \{p(n) \in GP_r : \{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_r, \forall \epsilon > 0\}.$$

Remark 7.4. It is clear that for $p(n) \in GP_r$, $p(n) \in \widehat{GP}_r$ if and only if $-p(n) \in \widehat{GP}_r$. Since $\mathcal{N}_r$ is a filter, if $p_1(n), p_2(n), \cdots, p_k(n) \in \widehat{GP}_r$ then

$$p_1(n) + p_2(n) + \cdots + p_k(n) \in \widehat{GP}_r.$$

Moreover by the definition of $\widehat{GP}_r$, we know that $\mathcal{F}_{GP_r} \subset \mathcal{N}_r$ if and only if $\widehat{GP}_r = GP_r$.

Since we will use induction to show Theorem C, thus we need to obtain some results under the following assumption, that is for some $d \geq 2$,

$$\mathcal{F}_{GP_{d-1}} \subseteq \mathcal{N}_{d-1}. \tag{7.4}$$

Lemma 7.5. *Let $p(n), q(n) \in GP_d$ with $p(n) \simeq_d q(n)$. Under the assumption* (7.4)*, $p(n) \in \widehat{GP}_d$ if and only if $q(n) \in \widehat{GP}_d$.*

Proof. It follows from Lemma 6.3, $\mathcal{N}_d$ being a filter and $\mathcal{F}_{GP_{d-1}} \subseteq \mathcal{N}_{d-1} \subseteq \mathcal{N}_d$. □

Theorem 7.6. *Under the assumption* (7.4)*, the following properties are equivalent:*

(1) $\mathcal{F}_{GP_d} \subseteq \mathcal{N}_d$.

(2) $P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \in \widehat{GP}_d$ *for any* $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$, *that is*

$$\{n \in \mathbb{Z} : P(n; \alpha_1, \alpha_2, \cdots, \alpha_d) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_d$$

for any $\alpha_1, \alpha_2, \cdots, \alpha_d \in \mathbb{R}$ *and* $\epsilon > 0$.

(3) $SGP_d \subset \widehat{GP}_d$.

Proof. The proof is similar to that of Theorem 6.7. □

7.3. **Proofs of Theorem C.** Now we prove $\mathcal{F}_{GP_d} \subseteq \mathcal{N}_d$ by induction on d. When $d = 1$, since $\mathcal{F}_{GP_1} = \mathcal{F}_{SGP_1}$ and $\mathcal{N}_d$ is a filer, it is sufficient to show that: for any $p(n) = an \in SGP_1$ and $\epsilon > 0$, we have

$$\{n \in \mathbb{Z} : p(n) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \in \mathcal{N}_1. \tag{7.5}$$

This is easy to be verified.

Now we assume that for $d \geq 2$, $\mathcal{F}_{GP_{d-1}} \subseteq \mathcal{N}_{d-1}$, i.e. (7.4) holds. Then it follows from Theorem 7.6 that under the assumption (7.4), to show $\mathcal{F}_{GP_d} \subseteq \mathcal{N}_d$, it is sufficient to show that

$$P(n; \beta_1, \beta_2, \ldots, \beta_d) \in \widehat{GP}_d,$$

for any $\beta_1, \beta_2, \cdots, \beta_d \in \mathbb{R}$.

Fix $\beta_1, \beta_2, \ldots, \beta_d \in \mathbb{R}$. We divide the remainder of the proof into two steps.

Step 1. We are going to show

$$P(n;\beta_1,\beta_2,\cdots,\beta_d)\simeq_d \sum_{\ell=1}^{d}\sum_{\substack{j_1,\cdots j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=d}}(-1)^{\ell-1}\lambda U(n;j_1,j_2,\cdots,j_\ell),$$

where as in the proof of Theorem B, we define

$$U(n;j_1)=\frac{n^{j_1}}{j_1!}\prod_{r=1}^{j_1}\alpha_r,\ 1\le j_1\le d. \tag{7.6}$$

And inductively for $\ell=2,3,\cdots,d$ define

$$\begin{aligned}U(n;j_1,j_2,\cdots,j_\ell)&=(U(n;j_1,\cdots,j_{\ell-1})-\lceil U(n;j_1,\cdots,j_{\ell-1})\rceil)\frac{n^{j_\ell}}{j_\ell!}\prod_{r=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r}\\ &=(U(n;j_1,\cdots,j_{\ell-1})-\lceil U(n;j_1,\cdots,j_{\ell-1})\rceil)L(\frac{n^{j_\ell}}{j_\ell!}\prod_{r_\ell=1}^{j_\ell}\alpha_{\sum_{t=1}^{\ell-1}j_t+r_\ell})\end{aligned}$$

for $j_1,j_2,\cdots,j_\ell\ge 1$ and $j_1+\cdots+j_\ell\le d$ (see (4.1) for the definition of L).

In fact, let $\lambda_1,\lambda_2,\ldots,\lambda_d\in\mathbb{Z}$ and $\lambda\in\mathbb{N}$ satisfying (7.3). Put

$$\alpha_1=\beta_1/\lambda,\alpha_2=\beta_2,\alpha_3=\beta_3,\ldots,\alpha_d=\beta_d.$$

Then

$$P(n;\beta_1,\beta_2,\ldots,\beta_d)=\lambda P(n;\alpha_1,\alpha_2,\ldots,\alpha_d).$$

Note that in proof of Theorem B we have

$$P(n;\alpha_1,\alpha_2,\cdots,\alpha_d)\simeq_d \sum_{\ell=1}^{d}\sum_{\substack{j_1,\cdots j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=d}}(-1)^{\ell-1}U(n;j_1,j_2,\cdots,j_\ell). \tag{7.7}$$

Since λ is an integer, we have

$$\lambda P(n;\alpha_1,\alpha_2,\cdots,\alpha_d)\simeq_d \sum_{\ell=1}^{d}\sum_{\substack{j_1,\cdots j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=d}}(-1)^{\ell-1}\lambda U(n;j_1,j_2,\cdots,j_\ell).$$

That is,

$$P(n;\beta_1,\beta_2,\cdots,\beta_d)\simeq_d \sum_{\ell=1}^{d}\sum_{\substack{j_1,\cdots j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=d}}(-1)^{\ell-1}\lambda U(n;j_1,j_2,\cdots,j_\ell).$$

Hence, by Lemma 7.5, to show $P(n;\beta_1,\beta_2,\cdots,\beta_d)\in\widehat{GP}_d$, it suffices to show

$$\sum_{\ell=1}^{d}\sum_{\substack{j_1,\cdots j_\ell\in\mathbb{N}\\ j_1+\cdots+j_\ell=d}}(-1)^{\ell-1}\lambda U(n;j_1,j_2,\cdots,j_\ell)\in\widehat{GP}_d. \tag{7.8}$$

Now choose $\mathbf{x} = (x_i^k)_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ with $x_i^1 = \alpha_i$ for $i = 1, 2, \cdots, d$ and $x_i^k = 0$ for $2 \le k \le d$ and $1 \le i \le d-k+1$. Let

$$A = \mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & \alpha_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & \alpha_{d-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha_d \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

For $n \in \mathbb{N}$, if $\mathbf{x}(n) = (x_i^k(n))_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$ satisfies $\mathbf{M}(\mathbf{x}(n)) = A^n$, then by Lemma 6.1 and Remark 6.2,

$$x_i^k(n) = \binom{n}{k} \alpha_i \alpha_{i+1} \cdots \alpha_{i+k-1} \tag{7.9}$$

for $1 \le k \le d$ and $1 \le i \le d-k+1$.

Let $X = \mathbb{G}_d/\Gamma$ be endowed with the metric ρ in Lemma 3.7 and T be the nil-rotation induced by $A \in \mathbb{G}_d$, i.e. $B\Gamma \mapsto AB\Gamma$ for $B \in \mathbb{G}_d$. Since $\mathbb{G}_d$ is a d-step nilpotent Lie group and Γ is a uniform subgroup of $\mathbb{G}_d$, (X, T) is a d-step nilsystem. Let $x_0 = \Gamma \in X$ and Z be the closure of the orbit $\mathrm{orb}(x_0, T)$ of e in X. Then (Z, T) is a minimal d-step nilsystem. We consider ρ as a metric on Z.

Step 2. For any $\epsilon > 0$, we are going to show there is a nonempty open set U of Z such that

$$\begin{aligned} &\{n \in \mathbb{Z} : \sum_{\ell=1}^{d} \sum_{\substack{j_1,\cdots j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \cdots, j_\ell) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\} \\ &\supseteq \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}. \end{aligned} \tag{7.10}$$

That means $\sum\limits_{\ell=1}^{d} \sum\limits_{\substack{j_1,\cdots j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \cdots, j_\ell) \in \widehat{GP}_d$.

Fix an $\epsilon > 0$. Take $\epsilon_2 = \min\Big\{\dfrac{\epsilon}{2K(\sum_{i=0}^{d-1} d^i)}, \dfrac{1}{4}\Big\}$, where $K = \sum\limits_{m=1}^{d} |\lambda_m| \Big(\sum\limits_{t=0}^{d} m^t\Big)$, and let $\epsilon_1 > 0$ be small enough such that $e^{\epsilon_1 + \epsilon_1^2 + \ldots + \epsilon_1^d} - 1 < \epsilon_2$. Let

$$U = \{z \in Z : \rho(z, x_0) < \epsilon_1\} = \{c\Gamma \in Z : \rho(c\Gamma, \Gamma) < \epsilon_1\}.$$

and let

$$S = \{n \in \mathbb{Z} : U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset\}.$$

Now we show that

$$S \subseteq \Big\{n \in \mathbb{Z} : \sum_{\ell=1}^{d} \sum_{\substack{j_1,\cdots j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \cdots, j_\ell) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon)\Big\}$$

Let $n \in S$. Then $U \cap T^{-n}U \cap \ldots \cap T^{-dn}U \neq \emptyset$. Hence there is some $B \in \mathbb{G}_d$ with

$$B\Gamma \in U \cap T^{-n}U \cap \ldots \cap T^{-dn}U.$$

Thus

$$\rho(A^{mn}B\Gamma, \Gamma) < \epsilon_1, \ m = 0, 1, 2, \dots, d-1.$$

Since ρ is right translation invariant, we may assume that $\rho(B, I) < \epsilon_1$, where I is the $(d+1) \times (d+1)$ identity matrix.

For each $m \in \{1, 2, \dots, d\}$, since $\rho(A^{mn}B\Gamma, \Gamma) < \epsilon_1$ there is some $C_m \in \Gamma$ such that

$$\rho(A^{mn}BC_m, I) < \epsilon_1. \tag{7.11}$$

Let $A^{mn}BC_m = \mathbf{M}(\mathbf{z}(m))$, where $\mathbf{z}(m) = (z_i^k(m))_{1\le k\le d, 1\le i\le d-k+1} \in \mathbb{R}^{d(d+1)/2}$. Then From (7.11), we have $||A^{mn}BC_m - I|| < \epsilon_2$ by Lemma 3.6, thus

$$|z_i^k(m)| < \epsilon_2, \quad 1 \le k \le d, 1 \le i \le d-k+1.$$

On the one hand, since $|z_1^d(m)| < \epsilon_2$, we have

$$\sum_{m=1}^{d} \lambda_m z_1^d(m) \in (-K\epsilon_2, K\epsilon_2). \tag{7.12}$$

On the other hand, we have

$$\begin{aligned}\sum_{m=1}^{d} \lambda_m z_1^d(m) \approx & \Big(\sum_{l=1}^{d} (-1)^{l-1} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1+j_2+\dots+j_l = d}} \lambda U(n; j_1, j_2, \dots, j_l) \Big) \\ & + \triangle \Big((d + d^2 + \dots + d^{d-1})(2K\epsilon_2) \Big).\end{aligned} \tag{7.13}$$

Note that for $a, b \in \mathbb{R}$ and $\delta > 0$, $a \approx b+ \triangle(\delta)$ means that $a - b \pmod{\mathbb{Z}} \in (-\delta, \delta)$.

Since the proof of (7.13) is long, we put it after Theorem C. Now we continue the proof. By (7.13) and (7.12), we have

$$\sum_{l=1}^{d} (-1)^{l-1} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1+j_2+\dots+j_l = d}} \lambda U(n; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in \Big(-M(2K\epsilon_2), M(2K\epsilon_2) \Big) \subseteq (-\epsilon, \epsilon),$$

where $M = 1 + d + \dots + d^{d-1}$. This means that

$$n \in \Big\{ q \in \mathbb{Z} : \sum_{l=1}^{d} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1+j_2+\dots+j_l = d}} (-1)^{l-1} \lambda U(q; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon) \Big\}.$$

Hence

$$S \subseteq \Big\{ q \in \mathbb{Z} : \sum_{l=1}^{d} \sum_{\substack{j_1, j_2, \dots, j_l \in \mathbb{N} \\ j_1+j_2+\dots+j_l = d}} (-1)^{l-1} \lambda U(q; j_1, j_2, \dots, j_l) \pmod{\mathbb{Z}} \in (-\epsilon, \epsilon) \Big\}.$$

Thus we have proved (7.10) which means $\sum_{\ell=1}^{d} \sum_{\substack{j_1, \cdots j_\ell \in \mathbb{N} \\ j_1+\cdots+j_\ell = d}} (-1)^{\ell-1} \lambda U(n; j_1, j_2, \cdots, j_\ell) \in \widehat{GP}_d$. The proof of Theorem C is now finished.

7.4. **Proof of (7.13).** Since $\rho(B,I)<\epsilon_1$, by Lemma 3.6,

$$||B-I||<e^{\epsilon_1+\epsilon_1^2+\ldots+\epsilon_1^d}-1<\epsilon_2<1/2. \tag{7.14}$$

Denote $B=\mathbf{M}(\mathbf{y})$, where $\mathbf{y}=(y_i^k)_{1\le k\le d,1\le i\le d-k+1}\in\mathbb{R}^{d(d+1)/2}$. From (7.14),

$$|y_i^k|<\epsilon_2,\quad 1\le k\le d,\ 1\le i\le d-k+1.$$

For $m=1,2,\cdots,m$, Recall that $C_m\in\Gamma$ satisfies

$$\rho(A^{mn}BC_m,I)<\epsilon_1. \tag{7.15}$$

Denote $C_m=\mathbf{M}(\mathbf{h}(m))$, where $\mathbf{h}(m)=(-h_i^k(m))_{1\le k\le d,1\le i\le d-k+1}\in\mathbb{Z}^{d(d+1)/2}$. From (7.15), we have

$$||A^{mn}BC_m-I||<\epsilon_2,\ m=1,2,\ldots,d.$$

Let $A^{mn}B=\mathbf{M}(\mathbf{w}(m))$, where $\mathbf{w}(m)=(w_i^k(m))_{1\le k\le d,1\le i\le d-k+1}\in\mathbb{R}^{d(d+1)/2}$. Then

$$\begin{aligned} w_i^k(m)&=x_i^k(mn)+\Big(\sum_{j=1}^{k-1}x_i^j(mn)y_{i+j}^{k-j}\Big)+y_i^k\\ &=\tbinom{mn}{k}\alpha_i\alpha_{i+1}\cdots\alpha_{i+k-1}+\sum_{j=1}^{k-1}\tbinom{mn}{j}\alpha_i\alpha_{i+1}\cdots\alpha_{i+j-1}y_{i+j}^{k-j}+y_i^k \qquad (7.16)\\ &\triangleq\frac{(mn)^k}{k!}\alpha_i\ldots\alpha_{i+k-1}+\sum_{j=1}^{k-1}m^ja_i^k(j)+a_i^k(0),\end{aligned}$$

where $m=1,2,\ldots,d$, $a_i^k(j)$ does not depend on m and $|a_i^k(0)|=|y_i^k|<\epsilon_2$.

Recall that $\mathbf{z}(m)=(z_i^k(m))_{1\le k\le d,1\le i\le d-k+1}\in\mathbb{R}^{d(d+1)/2}$ satisfies $A^{mn}BC_m=\mathbf{M}(\mathbf{z}(m))$. Hence

$$z_i^k(m)=w_i^k(m)-\Big(\sum_{j=1}^{k-1}w_i^j(m)h_{i+j}^{k-j}(m)\Big)-h_i^k(m). \tag{7.17}$$

From $||A^{mn}BC_m-I||<\epsilon_2$, we have

$$|z_i^k(m)|<\epsilon_2,\quad 1\le k\le d,1\le i\le d-k+1.$$

Note that $h_i^k(m)\in\mathbb{Z}$, and we have

$$h_i^k(m)=\Big\lceil w_i^k(m)-\sum_{j=1}^{k-1}w_i^j(m)h_{i+j}^{k-j}(m)\Big\rfloor.$$

Let

$$u_i^k(m)=w_i^k(m)-\sum_{j=1}^{k-1}w_i^j(m)h_{i+j}^{k-j}(m).$$

Then

$$|u_i^k(m)-h_i^k(m)|=|z_i^k(m)|<\epsilon_2<1/2.$$

Recall that for $a,b\in\mathbb{R}$ and $\delta>0$, $a\approx b+\triangle\ (\delta)$ means $a-b\pmod{\mathbb{Z}}\in(-\delta,\delta)$.

Claim: *Let $1 \leq r \leq d-1$ and $v_r(0), v_r(1), \ldots, v_r(r) \in \mathbb{R}$. Then for each $1 \leq r_1 \leq d-r-1$ and $1 \leq j \leq r_1 + r$, there exist $v_{r,r_1}(j) \in \mathbb{R}$ such that*

(1)

$$\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t v_r(t) \Big) h_{1+r}^{d-r}(m) \approx \lambda(v_r(r) - \lceil v_r(r) \rceil) \frac{n^{d-r}}{(d-r)!} \alpha_{1+r} \ldots \alpha_d - \sum_{r_1=1}^{d-r-1} \sum_{m=1}^{d} \lambda_m \Big(\sum_{t=1}^{r_1+r} m^t v_{r,r_1}(t) \Big) h_{1+r+r_1}^{d-r-r_1}(m) + \triangle\,(2K\epsilon_2)$$

(2) $v_{r,r_1}(r+r_1) = \Big(v_r - \lceil v_r \rceil \Big) \dfrac{n^{r_1}}{r_1!} \alpha_{r+1} \ldots \alpha_{r+r_1}$ *for all* $1 \leq r_1 \leq d-r-1$.

Proof of Claim. Since $|u_{1+r}^{d-r}(m) - h_{1+r}^{d-r}(m)| < \epsilon_2$, we have

$$\Big| \sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t (v_r(r) - \lceil v_r(r) \rceil) \Big) \Big| \leq \sum_{m=1}^{d} |\lambda_m| \Big(\sum_{t=0}^{r} m^t \Big) \leq \Big(\sum_{m=1}^{d} |\lambda_m| \Big) \Big(\sum_{t=0}^{r} m^t \Big) = K.$$

Hence

$$\begin{aligned} &\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t v_r(t) \Big) h_{1+r}^{d-r}(m) \\ \approx &\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t (v_r(t) - \lceil v_r(t) \rceil) \Big) h_{1+r}^{d-r}(m) \\ \approx &\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t (v_r(t) - \lceil v_r(t) \rceil) \Big) u_{1+r}^{d-r}(m) + \triangle\,(K\epsilon_2). \end{aligned} \tag{7.18}$$

Then we have

$$\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t (v_r(t) - \lceil v_r(t) \rceil) \Big) u_{1+r}^{d-r}(m)$$

$$\sum_{m=1}^{d} \lambda_m \Big(\sum_{t=0}^{r} m^t (v_r(t) - \lceil v_r(t) \rceil) \Big) \Big(w_{1+r}^{d-r}(m) - \sum_{r_1=1}^{d-r-1} w_{1+r}^{r_1}(m) h_{1+r+r_1}^{d-r-r_1}(m) \Big).$$

From (7.16) we have

$$\begin{aligned}
&\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)w_{1+r}^{d-r}(m)\\
&=\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)\Big(\frac{(mn)^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d+\sum_{j=0}^{d-r-1}m^j a_{1+r}^{d-r}(j)\Big)\\
&=\sum_{m=1}^{d}\lambda_m m^d\frac{n^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d\Big(v_r(t)-\lceil v_r(t)\rceil\Big)\\
&\qquad+\sum_{h=1}^{d-1}\sum_{m=1}^{d}\lambda_m m^h\Bigg(\sum_{\substack{0\le t\le r\\0\le j\le d-r-1\\t+j=h}}(v_r(t)-\lceil v_r(t)\rceil)a_{1+r}^{d-r}(j)\Bigg)\\
&\qquad+\sum_{m=1}^{d}\lambda_m\Big(v_r(0)-\lceil v_r(0)\rceil\Big)a_{1+r}^{d-r}(0),
\end{aligned}$$

and so $\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)w_{1+r}^{d-r}(m)$ is equal to

$$\begin{aligned}
&\lambda\frac{n^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d(v_r(t)-\lceil v_r(t)\rceil)+(\sum_{m=1}^{d}\lambda_m)(v_r(0)-\lceil v_r(0)\rceil)y_{1+r}^{d-r}\\
&\approx\lambda\frac{n^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d(v_r(t)-\lceil v_r(t)\rceil)+\vartriangle(K\epsilon_2).
\end{aligned}$$

The last equation follows from

$$\Big|(\sum_{m=1}^{d}\lambda_m)(v_r(0)-\lceil v_r(0)\rceil)y_{1+r}^{d-r}\Big|\le\sum_{m=1}^{d}|\lambda_m|\epsilon_2<K\epsilon_2.$$

Then for $1\le r_1\le d-r-1$, by (7.16), we have

$$\begin{aligned}
&\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)w_{1+r}^{r_1}(m)h_{1+r+r_1}^{d-r-r_1}(m)\\
&=\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)\Big(\frac{(mn)^{r_1}}{(r_1)!}\alpha_{1+r}\dots\alpha_{r+r_1}+\sum_{j=0}^{r_1-1}m^j a_{1+r}^{r_1}(j)\Big)h_{1+r+r_1}^{d-r-r_1}(m)\\
&=\sum_{m=1}^{d}\lambda_m\Big(m^{r+r_1}\frac{n^{r_1}}{(r_1)!}\alpha_{1+r}\dots\alpha_{r+r_1}(v_r(t)-\lceil v_r(t)\rceil)+\\
&\qquad\sum_{h=0}^{r+r_1-1}m^h\big(\sum_{\substack{0\le t\le r\\0\le j\le r_1-1\\t+j=h}}\big(v_r(t)-\lceil v_r(t)\rceil\big)a_{1+r}^{r_1}(j)\big)\Big)h_{1+r+r_1}^{d-r-r_1}(m).
\end{aligned}$$

Let

$$v_{r,r_1}(h)=\sum_{\substack{0\le t\le r\\ 0\le j\le r_1-1\\ t+j=h}}\big(v_r(t)-\lceil v_r(t)\rceil\big)a_{1+r}^{r_1}(j),\ \ 0\le h\le r+r_1-1;$$

$$v_{r,r_1}(r+r_1)=\frac{(n)^{r_1}}{(r_1)!}\alpha_{1+r}\dots\alpha_{r+r_1}\left(v_r(t)-\lceil v_r(t)\rceil\right);$$

$$v_{r,r_1}(0)=\left(v_r(0)-\lceil v_r(0)\rceil\right)a_{1+r}^{r_1}(0)=\left(v_r(0)-\lceil v_r(0)\rceil\right)y_{1+r}^{r_1}(0).$$

It is easy to see that $|v_{r,r_1}(0)|<\epsilon_2$. Then

$$\begin{aligned}&\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t(v_r(t)-\lceil v_r(t)\rceil)\Big)w_{1+r}^{r_1}(m)h_{1+r+r_1}^{d-r-r_1}(m)\\ &=\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r+r_1}m^tv_{r,r_1}(t)\Big)h_{1+r+r_1}^{d-r-r_1}(m).\end{aligned}$$

To sum up, we have

$$\begin{aligned}&\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^t\left(v_r(t)-\lceil v_r(t)\rceil\right)\Big)u_{1+r}^{d-r}(m)\\ &\approx\lambda\frac{(n)^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d(v_r(t)-\lceil v_r(t)\rceil)-\sum_{r_1=1}^{d-r-1}\left(\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r_1+r}m^tv_{r,r_1}(t)\Big)h_{1+r+r_1}^{d-(r+r_1)}(m)\right)\\ &\qquad\qquad+\triangle\ (K\epsilon_2).\end{aligned}$$

Together with (7.18), we have

$$\begin{aligned}\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r}m^tv_r(t)\Big)h_{1+r}^{d-r}(m)&\approx\lambda\frac{(n)^{d-r}}{(d-r)!}\alpha_{1+r}\dots\alpha_d\Big(v_r(t)-\lceil v_r(t)\rceil\Big)\\ &-\sum_{r_1=1}^{d-r-1}\left(\sum_{m=1}^{d}\lambda_m\Big(\sum_{t=0}^{r_1+r}m^tv_{r,r_1}(t)\Big)h_{1+r+r_1}^{d-(r+r_1)}(m)\right)+\triangle\ (2K\epsilon_2).\end{aligned}$$

The proof of the claim is completed. □

We will use the claim repeatedly. First using (7.17) we have

$$\sum_{m=1}^{d}\lambda_mz_1^d(m)\approx\sum_{m=1}^{d}\lambda_m\left(w_1^d(m)-\sum_{j_1=1}^{d-1}w_1^{j_1}(m)h_{1+j_1}^{d-j_1}(m)\right).$$

By (7.16), we have

$$\begin{aligned}\sum_{m=1}^{d}\lambda_mw_1^d(m)&\approx\sum_{m=1}^{d}\lambda_mm^d\frac{n^d}{d!}\alpha_1\dots\alpha_d+\sum_{m=1}^{d}\lambda_my_1^d\\ &\approx\lambda\frac{n^d}{d!}\alpha_1\dots\alpha_d+\triangle\ (K\epsilon_2).\end{aligned}$$

Using this, (7.16) and the claim, we have

$$\sum_{m=1}^{d}\lambda_m z_1^d(m)\approx \lambda\frac{n^d}{d!}\alpha_1\dots\alpha_d-\sum_{m=1}^{d}\lambda_m\sum_{j_1=1}^{d-1}\Big(m^{j_1}\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}+\sum_{t=0}^{j_1-1}m^t a_1^{j_1}(t)\Big)h_{1+j_1}^{d-j_1}(m)+\vartriangle(K\epsilon_2)$$

$$\approx \lambda\frac{n^d}{d!}\alpha_1\dots\alpha_d-\left(\sum_{j_1=1}^{d-1}\lambda\frac{n^{d-j_1}}{(d-j_1)!}\alpha_{1+j_1}\dots\alpha_d\Big(\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}-\lceil\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}\rceil\Big)\right)$$

$$+\sum_{j_1=1}^{d-1}\sum_{j_2=1}^{d-j_1-1}\left(\sum_{m=1}^{d}\lambda_m\Big(m^{j_1+j_2}\Big(\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}-\lceil\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}\rceil\Big)\frac{n^{j_2}}{j_2!}\alpha_{1+j_1}\dots\alpha_{j_1+j_2}\right.$$

$$\left.+\sum_{t=0}^{j_1+j_2-1}m^t v_{j_1,j_2}(t)\Big)h_{1+j_1+j_2}^{d-(j_1+j_2)}(m)\right)+\vartriangle\big(\big(2(d-1)K+K\big)\epsilon_2\big).$$

Note that here we use $v_{j_1}(t)=a_1^{j_1}(t), t=0,1,\dots,j_1-1$ and $v_{j_1}(j_1)=\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}$.

Recall the definition of $U(\cdot)$:

$$\frac{n^d}{d!}\alpha_1\dots\alpha_d=U(n;d),$$

$$\Big(\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}-\lceil\frac{n^{j_1}}{j_1!}\alpha_1\dots\alpha_{j_1}\rceil\Big)\frac{n^{j_2}}{j_2!}\alpha_{1+j_1}\dots\alpha_{j_1+j_2}=U(n;j_1,j_2).$$

Substituting these in the above equation, we have

$$\sum_{m=1}^{d}\lambda_m z_1^d(m)\approx\lambda U(n;d)-\sum_{j_1=1}^{d-1}\lambda U(n;j_1,d-j_1)$$

$$+\sum_{j_1=1}^{d-1}\sum_{j_2=1}^{d-j_1-1}\left(\sum_{m=1}^{d}\lambda_m\Big(m^{j_1+j_2}U(n;j_1,j_2)+\sum_{t=0}^{j_1+j_2-1}m^t v_{j_1,j_2}(t)\Big)h_{1+j_1+j_2}^{d-(j_1+j_2)}(m)\right)$$

$$+\vartriangle(2dK\epsilon_2)$$

Using the claim again, we have:

$$\sum_{m=1}^{d}\lambda_m z_1^d(m)\approx\lambda U(n;d)-\sum_{j_1=1}^{d-1}\lambda U(n;j_1,j_2)+\sum_{j_1=1}^{d-1}\sum_{j_2=1}^{d-j_1-1}\lambda U(n;j_1,j_2,d-j_1-j_2)$$

$$-\sum_{j_1=1}^{d-1}\sum_{j_2=1}^{d-j_1-1}\sum_{j_3=1}^{d-(j_1+j_2)-1}\left(\sum_{m=1}^{d}\lambda_m\Big(m^{j_1+j_2+j_3}U(n;j_1,j_2,j_3)+\right.$$

$$\left.\sum_{t=0}^{j_1+j_2+j_3-1}m^t v_{j_1,j_2,j_3}(t)\Big)h_{1+j_1+j_2+j_3}^{d-(j_1+j_2+j_3)}(m)\right)+\vartriangle(2dK\epsilon_2+2d^2K\epsilon_2).$$

Inductively, we have $\sum_{m=1}^{d}\lambda_m z_1^d(m)$

$$\approx\Big(\sum_{l=1}^{d}(-1)^{l-1}\sum_{\substack{j_1,j_2,\dots,j_l\in\mathbb{N}\\ j_1+j_2+\dots+j_l=d}}\lambda U(n;j_1,j_2,\dots,j_l)\Big)+\vartriangle(2dK\epsilon_2+2d^2K\epsilon_2+\dots+2d^{d-1}K\epsilon_2)$$

$$\approx\Big(\sum_{l=1}^{d}(-1)^{l-1}\sum_{\substack{j_1,j_2,\dots,j_l\in\mathbb{N}\\ j_1+j_2+\dots+j_l=d}}\lambda U(n;j_1,j_2,\dots,j_l)\Big)+\vartriangle\Big((d+d^2+\dots+d^{d-1})(2K\epsilon_2)\Big).$$

The proof of (7.13) is now finished.

8. Applications

Our main results can be applied to get results in the theory of dynamical systems. As the limitation of the length of the paper, here we only state the results and the detailed proofs will appear in a forthcoming paper by the same authors.

8.1. **d-step almost automorpy.** The notion of almost automorphy was first introduced by Bochner in 1955 in a work of differential geometry [7, 8]. Veech showed that each almost automorphic minimal system is an almost one-to-one extension of a compact metric abelian group rotation [28]. Let (X,T) be a minimal system and $d \in \mathbb{N}$. (X,T) is called a *d-step almost automorphic* system if it is an almost one-to-one extension of a d-step nilsystem. Let $\pi : X \longrightarrow Y$ be the almost one-to-one extension with Y being a d-step nilsystem. A point $x \in X$ is *d-step almost automorphic* if $\pi^{-1}\pi(x) = \{x\}$.

Using the main results of this paper, we show that $\mathcal{F}_{Poi_d}$, $\mathcal{F}_{Bir_d}$ and $\mathcal{F}_{d,0}$ can be used to characterize d-step almost automorphy, i.e. in some sense, $\mathcal{F}_{Poi_d}$ and $\mathcal{F}^*_{d,0}$ can not be distinguished "dynamically". Similar results can be found in the next subsections.

Theorem 8.1. [22] *Let (X,T) be a minimal system, $x \in X$ and $d \in \mathbb{N}$. Then the following statements are equivalent*

(1) *x is d-step almost automorphic.*
(2) *$N(x,V) \in \mathcal{F}^*_{Poi_d}$ for each neighborhood V of x.*
(3) *$N(x,V) \in \mathcal{F}^*_{Boi_d}$ for each neighborhood V of x.*
(4) *$N(x,V) \in \mathcal{F}_{d,0}$ for each neighborhood V of x.*

8.2. **Regionally proximal relation of order d.**

8.2.1. *Regionally proximal relation.* Regionally proximal relation plays a very import role in the theory of topological dynamics. It is the main tool to characterize the equicontinuous structure relation $S_{eq}(X)$ of a system (X,T); i.e. to find the smallest closed invariant equivalence relation $R(X)$ on (X,T) such that $(X/R(X),T)$ is equicontinuous. Veech [29] gave the first proof of the fact that the regionally proximal relation is an equivalence one. Also he showed that Poincaré sets can be used to characterize regionally proximal relation.

8.2.2. *Regionally proximal relation of order d.* In [17] Host and Kra defined a d-step nilfactor for each ergodic system, see also [31]. To get a similar factor in topological dynamics Host, Maass and Kra introduced the notion of regionally proximal relation of higher order.

Definition 8.2. [20, 19] Let (X,T) be a system and let $d \geq 1$ be an integer. A pair $(x,y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$, there exist $x', y' \in X$ and a vector $\mathbf{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0,1\}^d$ such that $d(x,x') < \delta, d(y,y') < \delta$, and

$$\rho(T^{\mathbf{n}\cdot\epsilon}x', T^{\mathbf{n}\cdot\epsilon}y') < \delta \text{ for any } \epsilon \neq (0,0,\ldots,0),$$

where $\mathbf{n}\cdot\epsilon = n_1\epsilon_1 + \ldots + n_d\epsilon_d$.

The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X)$, which is called *the regionally proximal relation of order d*.

It is easy to see that $\mathbf{RP}^{[d]}(X)$ is a closed and invariant relation for all $d \in \mathbb{N}$. When $d = 1$, $\mathbf{RP}^{[d]}(X)$ is nothing but the classical regionally proximal relation. In [19], for distal minimal systems the authors showed that $\mathbf{RP}^{[d]}(X)$ is a closed invariant equivalence relation, and the quotient of X under this relation is its maximal d-step nilfactor. Recently, these results are shown to be true for general minimal systems by Shao-Ye [27]. We remark that a point is d-step almost automorphic if and only if $\mathbf{RP}^{[d]}[x] = \{x\}$. Moreover, we have the following theorem whose proofs are based on the results built in this paper.

Theorem 8.3. [22] *Let (X, T) be a minimal system and $d \in \mathbb{N}$. The following statements are equivalent:*

(1) $(x, y) \in \mathbf{RP}^{[d]}(X)$
(2) *$N(x, U) \in \mathcal{F}_{Poi_d}$ for each neighborhood U of y.*
(3) *$N(x, U) \in \mathcal{F}_{Bir_d}$ for each neighborhood U of y.*
(4) *$N(x, U) \in \mathcal{F}^*_{d,0}$ for each neighborhood U of y.*

We remark that for $d = 1$ the above theorem is known, see for example [29, 9, 21].

8.3. **Nil$_d$ Bohr$_0$ sets and sets $SG_d(P)$.** In this article to study Nil$_d$ Bohr$_0$ sets, we use generalized polynomials. In [18], Host and Kra introduced an interesting set (SG_d set) to study the Nil$_d$ Bohr$_0$ sets. Here we state some results and give some questions. First we recall definitions introduced by Host and Kra in [18].

8.3.1. *Sets $SG_d(P)$ and SG^*_d.* Let $d \geq 0$ be an integer and let $P = \{p_i\}_i$ be a (finite or infinite) sequence in $\mathbb{N}$. The *set of sums with gaps of length less than d* of P is the set $SG_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \ldots + \epsilon_n p_n$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1 have length less than d. A subset $A \subseteq \mathbb{N}$ is an SG^*_d-set if $A \cap SG_d(P) \neq \emptyset$ for every infinite sequence P in $\mathbb{N}$.

Note that in this definition, P is a sequence and not a subset of $\mathbb{N}$. For example, if $P = \{p_1, p_2, \ldots\}$, then $SG_1(P)$ is the set of all sums $p_m + p_{m+1} + \ldots + p_n$ of consecutive elements of P, and thus it coincides with the set $\Delta(S)$ where $S = \{p_1, p_1 + p_2, p_1 + p_2 + p_3, \ldots\}$. Therefore SG^*_1-sets are the same as Δ^*-sets.

In [18] a notion called *strongly piecewise-F*, written PW- $\mathcal{F}$ was introduced and the following proposition was proved

Proposition 8.4. *Every SG^*_d-set is a PW-Nil$_d$ Bohr$_0$-set.*

Host and Kra asked the following question.

Question 8.5. *Is every Nil$_d$ Bohr$_0$-set an SG^*_d-set?*

Though we can not answer this question, we show that they can not be distinguished dynamically (see Theorems 8.3 and 8.7). Using Theorem B, Question 8.5 can be reformulated in the following way:

Question 8.6. *Let $d \in \mathbb{N}$ and S be an SG_d-set. Is it true that for any $k \in \mathbb{N}$, any $P_1, \dots, P_k \in \mathcal{F}_{SGP_d}$ and any $\epsilon_i > 0$, there is $n \in S$ such that*

$$P_i(n)(mod\ \mathbb{Z}) \in (-\epsilon_i, \epsilon_i)$$

for all $i = 1, \dots, k$?

We remark that since a d-step nilsystem is distal, the above question has an affirmative answer for any IP-set.

8.3.2. *SG_d sets and regionally proximal relation of order d.* We can use SG_d sets to characterize regionally proximal relation of order d. For each $d \in \mathbb{N}$, denote by SG_d the collection of all sets $SG_d(P)$, and $\mathcal{F}_{SG_d}$ the family generated by SG_d.

Theorem 8.7. [22] *Let (X, T) be a minimal system. Then for any $d \in \mathbb{N}$, $(x, y) \in \mathbf{RP}^{[d]}$ if and only if $N(x, U) \in \mathcal{F}_{SG_d}$ for each neighborhood U of y.*

A direct corollary of Theorem 8.7 is: let (X, T) be a minimal system, $x \in X$, and $d \in \mathbb{N}$. If x is $\mathcal{F}^*_{SG_d}$-recurrent, then it is d-step almost automorphic. Since $\mathcal{F}_{SG_d}$ does not have the Ramsey property [22], we do not know if the converse of the above corollary holds.

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