Generalized Measures for Fault Tolerance of Star Networks

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This article shows that, for any integers n and k with $0 \leq k \leq n-2$, at least $(k+1)!(n-k-1)$ vertices or edges have to be removed from an n-dimensional star graph to make it disconnected with no vertices of degree less than k. The result gives an affirmative answer to the conjecture proposed by Wan and Zhang (Appl Math Lett 22 (2009), 264-267). **NETWORKS, Vol. 63(3), 225–230 2014**

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1. INTRODUCTION

It is well known that interconnection networks play an important role in multiprocessor systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

A subset $S \subset V(G)$ (resp. $F \subset E(G)$) of a connected graph G is called a vertex-cut (resp. edge-cut) if $G - S$ (resp. $G - F$) is disconnected. The connectivity $\kappa(G)$ (resp. edge-connectivity $\lambda(G)$) of G is defined as the minimum cardinality over all vertex-cuts (resp. edge-cuts) of G. The connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ of a graph G are two important measurements for fault tolerance of the network since the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. However, in the definitions of $\kappa(G)$ and $\lambda(G)$, it is implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus they underestimate the resilience of the network. To overcome such shortcoming, Harary [6] introduced the concept of conditional connectivity by appending some requirements on the components of $G-S$ (resp. $G-F$). In this trend, Esfahanian [5] proposed the concept of restricted connectivity, Latifi et al. [8] generalized it to restricted k-connectivity which can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs.

Let G be a connected graph. A subset $S \subset V(G)$, if any, is called a k-vertex-cut, if $G - S$ is disconnected and has the minimum degree at least k. The *k-super connectivity* of G, denoted by $\kappa_s^{(k)}(G)$, is defined as the minimum cardinality over all k-vertex-cuts of G. Similarly, a subset $F \subset E(G)$, if any, is called a *k-edge-cut*, if $G-F$ is disconnected and has the minimum degree at least k. The *k-super edge-connectivity* of G, denoted by $\lambda_s^{(k)}(G)$, is defined as the minimum cardinality over all k-edge-cuts of G.

For an arbitrary connected graph G and an integer k, determining $\kappa_s^{(k)}(G)$ and $\lambda_s^{(k)}(G)$ is quite difficult, there is no known polynomial algorithm to compute them yet. In fact, for an arbitrarily given graph G and integer $k \geq 1$, the existence of $\kappa_s^{(k)}(G)$ and $\lambda_s^{(k)}(G)$ is an open problem so far. Only a little knowledge of results have been known on $\kappa_s^{(k)}$ and $\lambda_s^{(k)}$ for some special classes of graphs for any k. For example, for the hypercube Q_n, Oh et al. [12] and Wu et al. [17] independently determined $\kappa_s^{(k)}(Q_n) = 2^k(n-k)$ for $k \leq n-2$, Xu [18] determined $\lambda_s^{(k)}(Q_n) = 2^k(n-k)$ for $k \leq n-1$.

As an attractive alternative network to the hypercube, the n-dimensional star graph S_n is proposed by Akers et al. [1]. Since it has superior degree and diameter compared to the comparable hypercube as well as it is highly hierarchical and symmetrical [4], the star graph S_n has received considerable attention in recent years (see, e.g., [1, 3, 9, 10, 14–16]). In particular, Cheng and Lipman [2], Hu and Yang [7], Nie et al. [11], and Rouskov et al. [15], independently, determined

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$\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geq 3$. Yang et al. [20] proved $\lambda_s^{(2)}(S_n) = 6(n-3)$ for $n \geq 4$. Wan and Zhang [19] showed that $\kappa_s^{(2)}(S_n) = 6(n-3)$ for $n \geq 4$ and conjectured that $\kappa_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for $k \leq n-2$. In this article, we give an affirmative answer to the conjecture and generalize the afore mentioned results by proving that $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for any k with $0 \leq k \leq n-2$.

In section 2, we recall some structural properties of S_n and lemmas to be used in our proofs. The proofs of main results are in section 3. A conclusion is in section 4.

2. DEFINITIONS AND LEMMAS

For a given integer n with $n \geq 2$, let $I_n = \{1, 2, \ldots, n\}, I'_n = \{2, \ldots, n\}$ and $P(n) = \{p_1p_2 \ldots p_n : p_i \in I_n, p_i \neq p_j, 1 \leq i \neq j \leq n\}$, the set of permutations on I_n. Clearly, $|P(n)| = n!$. For a permutation $p = p_1 \ldots p_j \ldots p_n \in P(n)$, the digit p_j is called the symbol in the j-th position (or dimension) in p. For each $i \in I'_n$, we use p^i to denote the permutation obtained from p by exchanging two symbols in the first and the i-th position of p and leaving the rest unaltered, that is, $p^i = p_ip_2 \ldots p_{i-1}p_1p_{i+1} \ldots p_n$.

The n-dimensional star graph, denoted by S_n, is an undirected graph with vertex-set $P(n)$ and edge-set $\{pp^i : p \in P(n), i \in I'_n\}$. The star graphs S_2, S_3, and S_4 are shown in Figure 1.

Like the hypercube, the star graph is a vertex- and edge-transitive graph with degree $(n-1)$. Moreover, S_n is a Cayley graph on the symmetric group on I_n with respect to the generating set $\{t^2, t^3, \ldots, t^n\}$, where t is the identity permutation [1].

The following properties of S_n are very useful for our proofs.

Lemma 2.1 (see Cheng et al. [3], 2008). *If $n \geq 3$, then $\kappa(S_n) = \lambda(S_n) = n-1$, and the length of the shortest cycle in S_n is 6.*

For fixed $i, j \in I_n$, we use $S_n^{j:i}$ to denote the subgraph of S_n induced by all vertices with symbol i in the j-th position. From definition, it is easy to see that $S_n^{j:i}$ is isomorphic to S_{n-1} for each $i \in I_n$ and each $j \in I'_n$, and $S_n^{1:i}$ is an independent vertex set of size $(n-1)!$ for each $i \in I_n$.

Using these subgraphs yields two types of partitions for S_n according as the fixed index is i or j. If a dimension $j \in I'_n$ is fixed, then $\{S_n^{j:i} : i \in I_n\}$ is called the partition of S_n along the dimension j, or called the first partition for short. If a symbol $i \in I_n$ is fixed, then $\{S_n^{j:i} : j \in I_n\}$ is called the partition of S_n along the symbol i, or called the second partition for short. Figure 2 shows two types of partitions for S_4.

The following lemmas give two structural properties of S_n by using two partitions.

Lemma 2.2 (The first structural property, Akers and Krishnamurthy [1], 1989). *For a fixed dimension $j \in I'_n, S_n$ can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $i \in I_n$. Moreover, there are $(n-2)!$ independent edges between $S_n^{j:i_1}$ and $S_n^{j:i_2}$ for any $i_1, i_2 \in I_n$ with $i_1 \neq i_2$.*

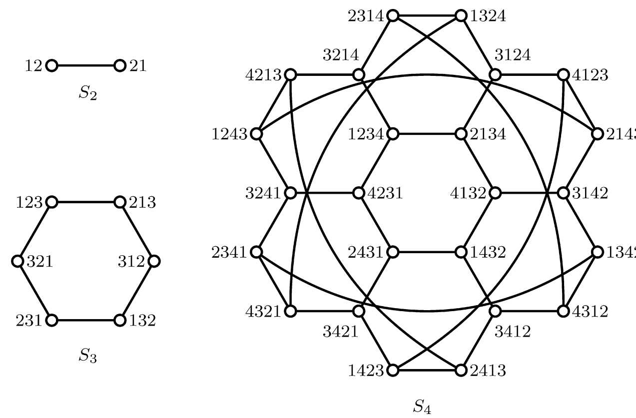

FIG. 1. The star graphs S_2, S_3 and S_4.

Lemma 2.3 (The second structural property, Shi et al. [13], 2012). *For a fixed symbol $i \in I_n, S_n$ can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an independent vertex set of size $(n-1)!$. Moreover, there are a perfect matching between $S_n^{1:i}$ and $S_n^{j:i}$ for any $j \in I'_n$, and there are no edges between $S_n^{j_1:i}$ and $S_n^{j_2:i}$ for any $j_1, j_2 \in I'_n$ with $j_1 \neq j_2$.*

3. MAIN RESULTS

In this section, we present our main results, that is, we determine the k-super connectivity and k-super edge-connectivity of the n-dimensional star graph S_n. We first investigate the properties of subgraph H of S_n with minimum degree $\delta(H)$ at least k. For a subset $X \subseteq V(S_n)$ and $j \in I_n$, we use U_j^X to denote the set of symbols in the j-th position of vertices in X, formally, $U_j^X = \{p_j : p_1 \ldots p_j \ldots p_n \in X\}$. The following lemma plays a key role in the proof of our main result.

Lemma 3.1. *Let H be a subgraph of S_n with vertex-set X. For a fixed $k \in I_{n-1}$, if $\delta(H) \geq k$, then there exists some $j \in I'_n$ such that $|U_j^X| \geq k+1$.*

Proof. Without loss of generality, we can assume that H is connected. For sake of simplicity, for a fixed X, we write U_j for U_j^X. Let W_i be the set of positions which symbol i appears in vertices in X excluding the first position, that is, $W_i = \{j \in I'_n : i \in U_j\}$.

We use the second partition of S_n to prove the lemma by induction on $n\ (\geq k+1)$.

If $n = k+1$, then $\delta(H) \geq k = n-1$, and so $H = S_n$. Since $|U_1| = \cdots = |U_n| = n = k+1$, the conclusion holds for $n = k+1$. We assume the conclusion is true for $n-1$ with $n \geq k+2$.

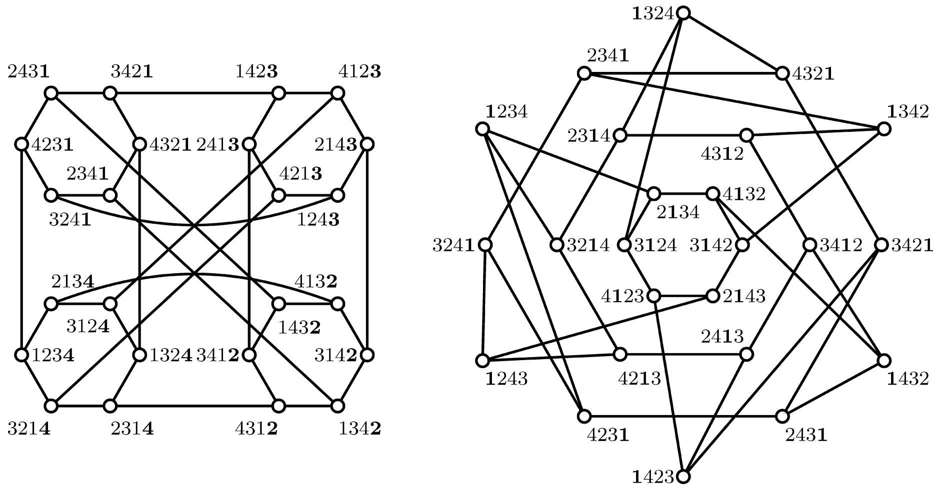

FIG. 2. Two perspectives of S_4, where the graph in the left-hand side is the first partition along the dimension 4, and one in the right-hand side is the second partition along the symbol 1.

Let $x = p_1p_2 \cdots p_n$ be a vertex in H. Then $x \in V(S_n^{1:p_1})$. By Lemma 2.3, all the neighbors of x are in different $S_n^{j:p_1}$ for each $j \in I'_n$. Since $\delta(H) \geq k, p_1$ appears in at least k different positions of vertices in H excluding the first position. It follows that

$$|W_{p_1}| \geq k \ \text{ for any } x = p_1p_2 \cdots p_n \in X. \tag{1}$$

If $|U_1| = n$, then each symbol of I_n appears in the first position of vertices in H. By (1), we have

$$|W_i| \geq k \ \text{ for each } i \in I_n. \tag{2}$$

Now we construct an $n \times (n-1)$ matrix $C = (c_{ij})_{n\times(n-1)}$, where c_{ij} is the indicator of whether i appears in position $j+1$ in the vertices of X, that is,

$$c_{ij} = \begin{cases} 1 & j+1 \in W_i\,; \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$|U_j| = \sum_{i=1}^{n} c_{i(j-1)} \ \text{ for each } j \in I'_n \text{ and}$$

$$|W_i| = \sum_{j=1}^{n-1} c_{ij} \ \text{ for each } i \in I_n.$$

It follows that

$$\sum_{j=2}^{n} |U_j| = \sum_{j=2}^{n}\sum_{i=1}^{n} c_{i(j-1)} = \sum_{i=1}^{n}\sum_{j=2}^{n} c_{i(j-1)}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n-1} c_{ij} = \sum_{i=1}^{n} |W_i|. \tag{3}$$

Combining (3) with (2), we have

$$\sum_{j=2}^{n} |U_j| = \sum_{i=1}^{n} |W_i| \geq nk. \tag{4}$$

If $|U_j| \leq k$ for each $j \in I'_n$, then $(n-1)k \geq nk$ by (4), a contradiction. Thus, there exists some $j \in I'_n$ such that $|U_j| \geq k+1$.

If $|U_1| < n$, then there exists at least one symbol in I_n that does not appear in the first position of any vertex in H. Without loss of generality, assume $1 \notin U_1$. Then $S_n^{1:1}$ does not contain vertices of H by the definition of U_1. By Lemma 2.3, H must be contained in the unique $S_n^{j_0:1}$ for some $j_0 \in I'_n$ since H is connected. Because $S_n^{j_0:1}$ is isomorphic to S_{n-1}, and $H \subseteq S_n^{j_0:1}$, by the induction hypothesis, there exists some $j \in I'_n$ such that $|U_j| \geq k+1$.

By the induction principle, the lemma follows. ■

Lemma 3.2. *For any integer k with $0 \leq k \leq n-2$, $\lambda_s^{(k)}(S_n) \leq (k+1)!(n-k-1)$ and $\kappa_s^{(k)}(S_n) \leq (k+1)!(n-k-1)$.*

Proof. Let X be the set of permutations on I_n whose the last $(n-k-1)$ positions is $12\cdots(n-k-1)$, and let H be the subgraph of S_n induced by X. Then, H is isomorphic to S_{k+1}. Let T be the set of neighbors of X in $S_n - X$ and F the set of edges between X and T in S_n. By the definition of S_n,

$$T = \{x^i : x \in X,\ i \in I_n \backslash I_{k+1}\}.$$

For a vertex of X, since it has k neighbors in X, it has exactly $(n-k-1)$ neighbors in T. In addition, it is easy to see that every vertex of T has exactly one neighbor in X. It follows that

$$|T| = |F| = (k+1)!(n-k-1).$$

Since every vertex v in $S_n - X$ has at most one neighbor in X and S_n is $(n-1)$-regular, v has at least $n-2$ neighbors in $S_n - X$, which implies that F is a k-edge-cut of S_n by $n-2 \geq k$ and the arbitrariness of v. It follows that

$$\lambda_s^{(k)}(S_n) \leq |F| = (k+1)!(n-k-1)$$

as desired, and so the first conclusion follows.

We now show that T is a k-vertex-cut of S_n. To this end, we only need to show that every vertex in $S_n - (X \cup T)$ has at least k neighbors within.

Let u be arbitrary vertex of $S_n - (X \cup T)$. We need to show that at most one of neighbors of u is in T. Suppose to the contrary that u has two distinct neighbors v and w in T. Then the first digits of v and w are different. Without loss of generality, assume $v = 1p_2 \dots p_{k+1}p_1 23 \cdots (n-k-1)$ and $w = 2p'_2 \dots p'_{k+1} 1p'_1 3 \cdots (n-k-1)$. Since u is adjacent to v, then u and v have exactly one digit difference excluding the first one. So are u and w. Therefore, w and v have exactly two digits difference excluding the first one. But w and v have two digits (the $(k+2)$-th and the $(k+3)$-th) difference, then $p_2 \dots p_{k+1} = p'_2 \dots p'_{k+1}$, therefore $p_1 = p'_1$, thus there exists a vertex $z = p_1 \cdots p_{k+1} 12 \cdots (n-k-1)$ in X such that $zv \in E(S_n)$ and $zw \in E(S_n)$, and so $zvuw$ is a cycle with length 4, which contradicts to the second conclusion in Lemma 2.1.

Since u has at most one neighbor in T, u has at least $(n-1)-1$ neighbors in $S_n - (X \cup T)$. Since $(n-1)-1 \geq k$, u has at least k neighbors in $S_n - (X \cup T)$, which implies that T is a k-vertex-cut of S_n. It follows that

$$\kappa_s^{(k)}(S_n) \leq |T| = (k+1)!(n-k-1)$$

as desired, and so the second conclusion follows. ∎

Theorem 3.3. $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ *for any k with* $0 \leq k \leq n-2$.

Proof. By Lemma 3.2, we only need to show that, for any k with $0 \leq k \leq n-2$,

$$\kappa_s^{(k)}(S_n) \geq (k+1)!(n-k-1) \text{ and } \lambda_s^{(k)}(S_n) \geq (k+1)!(n-k-1). \tag{5}$$

We prove (5) by induction on k. If $k=0$, then $\lambda_s^{(0)}(S_n) = \lambda(S_n) = n-1$ and $\kappa_s^{(0)}(S_n) = \kappa(S_n) = n-1$ by Lemma 2.1, and so (5) is true for $k=0$. Assume (5) holds for $k-1$ with $k \geq 1$, that is, for any k with $1 \leq k \leq n-2$,

$$\kappa_s^{(k-1)}(S_n) \geq k!(n-k) \text{ and } \lambda_s^{(k-1)}(S_n) \geq k!(n-k),$$

and so,

$$\kappa_s^{(k-1)}(S_{n-1}) \geq k!(n-k-1) \text{ and } \lambda_s^{(k-1)}(S_{n-1}) \geq k!(n-k-1). \tag{6}$$

Let T be a minimum k-vertex-cut (or k-edge-cut) of S_n. To prove (5), we only need to show that

$$|T| \geq (k+1)!(n-k-1) \text{ for } 1 \leq k \leq n-2. \tag{7}$$

To the end, let X be the vertex-set of a connected component H of $S_n - T$, and let

$$Y = \begin{cases} V(S_n - (X \cup T)) & \text{if } T \text{ is a vertex-cut;} \\ V(S_n - X) & \text{if } T \text{ is an edge-cut.} \end{cases}$$

Then $\delta(H) \geq k$, and so there exists some $j \in I'_n$ such that $|U_j^X| \geq k+1$ by Lemma 3.1. We choose $j_0 \in \{j \in I'_n : |U_j^X| \geq k+1\}$ such that $|U_{j_0}^X \cap U_{j_0}^Y| + |U_{j_0}^Y|$ is as large as possible. Without loss of generality, assume $j_0 = n$. In the following proof, we use the first partition of S_n. For $i \in I_n$, let

$$X_i = X \cap V(S_n^{n:i}), \quad Y_i = Y \cap V(S_n^{n:i}),$$
$$T_i = \begin{cases} T \cap V(S_n^{n:i}) & \text{if } T \text{ is a vertex-cut;} \\ T \cap E(S_n^{n:i}) & \text{if } T \text{ is an edge-cut,} \end{cases}$$

and let

$$J_X = \{i \in I_n : X_i \neq \emptyset\},$$
$$J_Y = \{i \in I_n : Y_i \neq \emptyset\}, \ J_0 = J_X \cap J_Y.$$

Clearly, $|J_X| = |U_n^X|$, $|J_Y| = |U_n^Y|$ and $|J_0| = |U_n^X \cap U_n^Y|$.

If $i \in J_0$, T_i is a vertex-cut (or an edge-cut) of $S_n^{n:i}$. For any vertex x in $S_n^{n:i} - T_i$, since x has degree at least k in $S_n - T$ and has exactly one neighbor outsider $S_n^{n:i}$, x has degree at least $k-1$ in $S_n^{n:i} - T_i$. Therefore, T_i is a $(k-1)$-vertex-cut (or a $(k-1)$-edge-cut) of $S_n^{n:i}$ for any $i \in J_0$. By the induction hypothesis (6), we have

$$|T_i| \geq k!(n-k-1) \text{ for each } i \in J_0. \tag{8}$$

If $|J_0| \geq k+1$, by (8) we have

$$|T| \geq \sum_{i=1}^{n} |T_i| \geq \sum_{i \in J_0} |T_i| \geq (k+1)k!(n-k-1)$$
$$= (k+1)!(n-k-1),$$

and so (7) follows.

Now assume $|J_0| \leq k$. Then $J_X \backslash J_0 \neq \emptyset$. We consider two cases, $J_Y \backslash J_0 \neq \emptyset$ and $J_Y \backslash J_0 = \emptyset$, respectively.

CASE 1. $J_Y \backslash J_0 \neq \emptyset$,

Let $E^{j_1 j_2}$ denote the set of edges between $S_n^{n:j_1}$ and $S_n^{n:j_2}$, and let

$$E_c = \{e \in E^{j_1 j_2} : j_1, j_2 \in I_n, \ j_1 \neq j_2\} \text{ and}$$
$$T_c = \begin{cases} \emptyset & \text{if } T \text{ is a vertex-cut;} \\ T \cap E_c & \text{if } T \text{ is an edge-cut.} \end{cases}$$

Assume $j_1 \in J_X \backslash J_0$, $j_2 \in J_Y \backslash J_0$. Then there are $(n-2)!$ independent edges between $S_n^{n:j_1}$ and $S_n^{n:j_2}$ by Lemma 2.2. Since each vertex in $S_n^{n:j_1}$ has a unique external neighbor, thus there are $(|J_X \backslash J_0||J_Y \backslash J_0|(n-2)!)$ independent edges between $\cup_{j_1 \in J_X \backslash J_0} S_n^{n:j_1}$ and $\cup_{j_2 \in J_Y \backslash J_0} S_n^{n:j_2}$. Note that each edge of these independent edges must have one end-vertex in T if T is a vertex-cut, and be contained in T_c if T is an edge-cut. Therefore, no matter whether T is a vertex-cut or an edge-cut, we have

$$\sum_{i \in (J_X \cup J_Y) \backslash J_0} |T_i| + |T_c| \geq |J_X \backslash J_0||J_Y \backslash J_0|(n-2)!. \tag{9}$$

Let

$$a = |J_X \backslash J_0|,\ b = |J_Y \backslash J_0|,\ c = |I_n \backslash (J_X \cup J_Y)|.$$

Then $a \geq 1, b \geq 1, a+b+c = n - |J_0|$, and so

$$\begin{aligned} ab + c &= ab + (n - |J_0|) - (a+b) \\ &= (n - |J_0|) + (a-1)(b-1) - 1 \\ &\geq (n - |J_0| - 1), \end{aligned}$$

that is,

$$ab + c \geq (n - |J_0| - 1). \tag{10}$$

Note that $c = 0$ if T is an edge-cut. Thus if there exists some $i \in I_n \backslash (J_X \cup J_Y)$, then T is a vertex-cut and $T_i = S_n^{n:i}$, and so

$$|T_i| = (n-1)! \ \text{ if } i \in I_n \backslash (J_X \cup J_Y). \tag{11}$$

Combining (8), (9), and (11) with (10), we have that

$$\begin{aligned} |T| &= \sum_{i=1}^{n} |T_i| + |T_c| \\ &= \sum_{i \in J_0} |T_i| + \left(\sum_{i \in (J_X \cup J_Y) \backslash J_0} |T_i| + |T_c| \right) + \sum_{i \in I_n \backslash (J_X \cup J_Y)} |T_i| \\ &\geq |J_0| k!(n-k-1) + |J_X \backslash J_0||J_Y \backslash J_0|(n-2)! + c(n-1)! \\ &= |J_0| k!(n-k-1) + ab(n-2)! + c(n-1)! \\ &\geq |J_0| k!(n-k-1) + (ab+c)(n-2)! \\ &\geq |J_0| k!(n-k-1) + (n - |J_0| - 1)(n-2)! \\ &\geq (n-1)k!(n-k-1) \\ &\geq (k+1)!(n-k-1), \end{aligned}$$

and so (7) follows.

CASE 2. $J_Y \backslash J_0 = \emptyset$,

In this case $J_Y = J_0$, then $|U_n^Y| = |J_Y| \leq k$. Let $\overline{X}_i = V(S_n^{n:i}) \backslash X_i$ for each $i \in I_n \backslash J_0$. Note that for each $i \in I_n \backslash J_0, \overline{X}_i = T_i$ if T is a vertex-cut, and $\overline{X}_i = \emptyset$ if T is an edge-cut.

We first show that $|\overline{X}_i| \geq (n-2)!$ for any $i \in I_n \backslash J_0$. Suppose to the contrary that there exists some $i \in I_n \backslash J_0$ such that $|\overline{X}_i| < (n-2)!$.

We show $|U_j^{X_i}| \geq n-1$ for any $j \in I'_{n-1}$. On the contrary, there exists some $j \in I'_{n-1}$ such that $|U_j^{X_i}| \leq n-2$. Notice that the rightmost digit of every vertex in X_i is i. There is at least one symbol $i_1 \in I_n \backslash \{i\}$ that does not appear in the j-th position of any vertex in X_i. Thus, the vertices with symbol i_1 in the j-th position and symbol i in the n-th position are not contained in X_i, which means that $\overline{X}_i$ contains at least $(n-2)!$ vertices, that is, $|\overline{X}_i| \geq (n-2)!$, a contradiction. Thus, $|U_j^{X_i}| \geq n-1$, and so $|U_j^X| \geq n-1$ for any $j \in I'_{n-1}$.

Since $|U_n^Y| \leq k$ and the subgraph induced by Y has minimum degree at least k, by Lemma 3.1 there exists some $j_1 \in I'_{n-1}$ such that $|U_{j_1}^Y| \geq k+1$. Then $|U_{j_1}^X| \geq n-1$ and $|U_{j_1}^Y| \geq k+1$, and so $|U_{j_1}^X \cap U_{j_1}^Y| \geq k$, therefore $|U_{j_1}^X \cap U_{j_1}^Y| + |U_{j_1}^Y| \geq 2k+1$. Noting that $|U_n^X \cap U_n^Y| = |J_0| \leq k$ and $|U_n^Y| = |J_Y| = |J_0| \leq k$, we have that

$$|U_n^X \cap U_n^Y| + |U_n^Y| \leq 2k < 2k+1 \leq |U_{j_1}^X \cap U_{j_1}^Y| + |U_{j_1}^Y|.$$

However, this fact contradicts the choice of j_0 that $|U_{j_0}^X \cap U_{j_0}^Y| + |U_{j_0}^Y|$ is as large as possible since we have supposed that $j_0 = n$.

Thus, $|\overline{X}_i| \geq (n-2)!$ for any $i \in I_n \backslash J_0$. If T is an edge-cut, then $\overline{X}_i = \emptyset$, a contradiction. Therefore, T is a vertex-cut, and so $\overline{X}_i = T_i$. It follows that

$$|T_i| = |\overline{X}_i| \geq (n-2)! \ \text{ for each } i \in I_n \backslash J_0. \tag{12}$$

Combining (12) with (8), we have

$$\begin{aligned} |T| &= \sum_{i=1}^{n} |T_i| = \sum_{i \in J_0} |T_i| + \sum_{i \in I_n \backslash J_0} |T_i| \\ &\geq |J_0| k!(n-k-1) + (n - |J_0|)(n-2)! \\ &\geq (k+1)!(n-k-1). \end{aligned}$$

By induction principles, (7) holds and so the theorem follows. ■

Corollary 3.4 ([19,20]). $\kappa_s^{(2)}(S_n) = \lambda_s^{(2)}(S_n) = 6(n-3)$ *for* $n \geq 4$.

4. CONCLUSIONS

In this article, we consider the generalized measures of fault tolerance for networks, called the k-super connectivity $\kappa_s^{(k)}$ and the k-super edge-connectivity $\lambda_s^{(k)}$. For n-dimensional star graph S_n, which is an attractive alternative network to hypercubes, we prove that $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for $0 \leq k \leq n-2$, which gives an affirmative answer to the conjecture proposed by Wan and Zhang [19]. The results show that at least $(k+1)!(n-k-1)$ vertices or edges have to be removed from S_n to make it disconnected without vertices of degree less than k. Thus these results can provide more accurate measurements for fault tolerance of the system when n-dimensional star graphs is used to model the topological structure of a large-scale parallel processing system.

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