· QuickSort平均复杂度分析

$x_1 x_2 \ldots x_n$ 排序

partition mode: $\underbrace{\boxed{\qquad}}_{k-1}$ $X[k]$ $\underbrace{\boxed{\qquad}}_{n-k}$

$X[k]$被选取概率为 $\frac{1}{n}$，即第$k$个位置被选中为划分之位置

$C_n$表示size为$n$时的比较次数

$$C_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(C_{k-1} + C_{n-k})$$

$$= (n-1) + \frac{1}{n}\sum_{k=1}^{n} C_{k-1} + \frac{1}{n}\sum_{k=1}^{n} C_{n-k}$$

$$= (n-1) + \frac{1}{n}\sum_{k=0}^{n-1} C_k + \frac{1}{n}\sum_{k=0}^{n-1} C_k$$

$$= (n-1) + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

$$\Longrightarrow nC_n = n(n-1) + 2\sum_{k=0}^{n-1} C_k \quad \cdots\cdots ①$$

用$(n-1)$代入$n$ $\Longrightarrow$ $(n-1)C_{n-1} = (n-1)(n-2) + 2\sum_{k=0}^{n-2} C_k \quad \cdots ②$

$①-②$ $\Longrightarrow$ $nC_n - (n-1)C_{n-1} = 2(n-1) + 2C_{n-1}$

$$\Longrightarrow nC_n = 2(n-1) + (n+1)C_{n-1}$$

$$\Longrightarrow \frac{C_n}{n+1} = \frac{C_{n-1}}{n} + \frac{2(n-1)}{n(n+1)}$$

令 $D_n = \frac{C_n}{n+1}$, $\Longrightarrow D_n = D_{n-1} + \frac{2(n-1)}{n(n+1)}$

$$\Rightarrow D_n = 2\sum_{j=1}^{n}\frac{j-1}{j(j+1)} \qquad // \ \frac{2}{j+1} - \frac{1}{j} = \frac{j-1}{j(j+1)}$$

$$= 2\sum_{j=1}^{n}\frac{2}{j+1} - 2\sum_{j=1}^{n}\frac{1}{j}$$

$$= 4\sum_{j=2}^{n+1}\frac{1}{j} - 2\sum_{j=1}^{n}\frac{1}{j}$$

$$= 2\sum_{j=1}^{n}\frac{1}{j} + \frac{4}{n+1} - 4$$

$$= 2H_n - \frac{4n}{n+1} = 2\ln n + O(1) = \frac{2}{\log e}\log n + O(1)$$

$$\approx 1.44\log n$$

$$\therefore\ C_n = (n+1)D_n \approx 1.44\, n\log n$$